NPDE - TCA

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Finite Element Method (FEM)

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differential
operator

Many DE don't have analytical soln ⇒ find approx. soln

Ritz method — Galerkin method

★ Ritz method

variational method in which B.V.P. formulated in terms of variational expression - functional.

↑

its minimal — Given D.E. with boundary condn.

Define: $< \phi, \psi > = \int_{\Omega} \phi \psi^* d\Omega$

↑ complex conj.

Observe: $\underbrace{< L\phi, \psi > = < \phi, L\psi >}_{L \text{ is self-adjoint} \ — ?}$

$\int_{\Omega} L\phi \psi^* d\Omega$

and

$$< L\phi, \phi > = \begin{cases} > 0 , & \phi \neq 0 \\ = 0 , & \phi = 0. \end{cases}$$

Functional:

$$F(\tilde{\phi}) = \frac{1}{2} < L\tilde{\phi}, \tilde{\phi} > - \frac{1}{2} < \tilde{\phi}, f > - \frac{1}{2} < f, \tilde{\phi} >$$

↑ trial function

Suppose problem is real valued.

$$\tilde{\phi} \simeq \sum_{j=1}^{N} c_j v_j = \{c\}^T \{v\} = \{v\}^T \{c\}$$

↑ expansion functions (chosen)

↑ constant coff (to be determined)

$$\Rightarrow F(\tilde{\phi}) = \frac{1}{2} \int L(\{c\}^T\{v\})[\{c\}^T\{v\}]^* d\Omega - \frac{1}{2} \{c\}^T \int \{v\} f^* d\Omega - \frac{1}{2} \int f [\{c\}^T \{v\}]^* d\Omega$$

$$= \frac{1}{2} \{c\}^T \int_{\Omega} \{v\} L \{v\}^T d\Omega \{c\} - \{c\}^T \int_{\Omega} \{v\} f \, d\Omega$$

To minimise $F(\tilde{\phi}) \Rightarrow \frac{\partial F}{\partial c_i} = 0$

ie. $\frac{1}{2}\int_\Omega v_i \, L \{v\}^T \, d\Omega \, \{c\} + \frac{1}{2}\{c\}^T \int_\Omega \{v\} \, L v_i \, d\Omega - \int_\Omega v_i f \, d$

$$\Rightarrow \frac{\partial F}{\partial c_i} = \frac{1}{2} \sum_{j=1}^{N} c_j \int_\Omega (v_i L v_j + v_j L v_i) \, d\Omega - \int_\Omega v_i f \, d\Omega$$

$$= 0$$

$$\Rightarrow \frac{1}{2} \sum_{j=1}^{N} c_j \int_\Omega (v_i L v_j + v_j L v_i) \, d\Omega = \int_\Omega v_i f \, d\Omega$$

$$\Rightarrow [S]\{c\} = \{b\} \ldots$$

where $S_{ij} = \frac{1}{2}\int_\Omega (v_i L v_j + v_j L v_i) \, d\Omega$

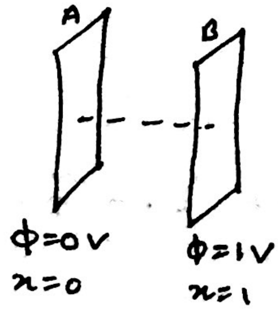
symm.

$b_i = \int_\Omega v_i f \, d\Omega$

ie. $S_{ij} = S_{ji}$

$= \int_\Omega v_i L v_j \, d\Omega$

example:



$\rho(x) = -(x+1)\varepsilon$

Using Poisson eq^n

$$\frac{\partial^2 \phi}{\partial x^2} = -\frac{\rho}{\varepsilon} = (x+1)$$

B.C.: $\phi|_{x=0} = 0$

$\phi|_{x=1} = 1$

For now. construct the functional: for given B.C. (taught later)

$$F(\tilde{\phi}) = \frac{1}{2}\int_0^1 \left(\frac{\partial \tilde{\phi}}{\partial x}\right)^2 dx + \int_0^1 (x+1)\,\tilde{\phi} \, dx. \quad ..①$$

$\tilde{\phi}_0 = \frac{\partial \tilde{\phi}}{\partial x}$?

$f = (x+1)$

<u>Proof/verification</u>

$$\tilde{\phi} = \underbrace{\phi(x)}_{\text{func. corresp to minimal of } F} + \underbrace{\delta\phi}_{\text{small \& arbitrary fn}}$$

$\delta\phi = 0$ at $\underbrace{x = 0, 1}_{\text{boundaries}}$

$$\Rightarrow \Delta F = F(\tilde{\phi}(x)) - F(\phi(x)) = \underbrace{\delta F}_{\text{first order in } \delta\phi} + \underbrace{O((\delta\phi)^2)}_{\text{contains higher orders of } \delta\phi}$$

~~$\delta\phi$~~ $\delta F \equiv$ first variation of $F = \lim_{e \to 0} \dfrac{F(\phi + e\,\delta\phi) - F(\phi)}{e}$

For F to be minimum $\tilde{\phi}(x) = \phi(x) \Rightarrow \delta F(\phi) = 0$

$$\delta F = \underbrace{\int_0^1 \frac{\partial\phi}{\partial x} \frac{\partial(\delta\phi)}{dx}\,dx}_{} + \int_0^1 (x+1)\,\delta\phi\,dx$$

$(\because \delta\phi|_{x=0,1} = 0$

$$\Rightarrow \delta F \doteq \cancelto{0}{\delta\phi \frac{d\phi}{dx}\Big|_{x=0}^{x=1}} - \int_0^1 \frac{\partial^2\phi}{\partial x^2}\,\delta\phi\,dx + \int_0^1 (x+1)\,\delta\phi\,dx$$

$$\Rightarrow \delta F = \int_0^1 \underbrace{\left(\frac{\partial^2\phi}{\partial x^2} - x - 1\right)}_{\text{given D.E.}} \delta\phi\,dx = 0$$

$\Rightarrow \phi(x)$ must satisfy D.E.

Lets take trial func. $\tilde{\phi}(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3$

Apply B.C. : $\tilde{\phi}(0) = 0 = c_1$

$\tilde{\phi}(1) = c_2 + c_3 + c_4 = 1$

$$\Rightarrow \tilde{\phi}(x) = x + c_3(x^2 - x) + c_4(x^3 - x)$$

Put in ①:

$$F = \frac{2}{5}c_4^2 + \frac{1}{6}c_3^2 + \frac{1}{2}c_3 c_4 - \frac{23}{60}c_4 - \frac{1}{4}c_3 + \frac{4}{3}$$

Minimisation

$$\left.\begin{aligned} \frac{\partial F}{\partial c_3} = 0 &\Rightarrow \frac{1}{3}c_3 + \frac{1}{2}c_4 - \frac{1}{4} = 0 \\ \frac{\partial F}{\partial c_4} = 0 &\Rightarrow \frac{1}{2}c_3 + \frac{4}{5}c_4 - \frac{23}{60} = 0 \end{aligned}\right\} c_3 = \frac{1}{2},\ c_4 = \frac{1}{6}$$

$$\Rightarrow \boxed{\phi(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{3}x}$$

★ Galerkin's method

weighted residual method.

Assume $\tilde{\phi}$ is approxi. solution to $L\phi = f$

residual $r = L\tilde{\phi} - f \neq 0$

reduce / minimise for all pts of Ω.

$$\Rightarrow \underset{\substack{\uparrow \\ \text{weighted} \\ \text{residual} \\ \text{integral}}}{R_i} = \int_{\Omega} \underset{\substack{\uparrow \\ \text{chosen} \\ \text{weighting} \\ \text{functions}}}{w_i}\, r \, d\Omega = 0$$

NOTE: weighting functions ≡ expansion of approx. solⁿ

like before, solⁿ $\tilde{\phi} = \sum_{j=1}^{N} c_j v_j = \{c\}^T \{v\}$

$$\Rightarrow w_i = v_i$$

$$\Rightarrow R_i = \int_{\Omega} (v_i L \{v\}^T \{c\} - v_i f) \, d\Omega = 0$$

$$\Rightarrow [S]\{c\} = \{b\}$$

where $S_{ij} = \int_{\Omega} v_i L v_j \, d\Omega$ — not necess. symmetric

$b = \int_{\Omega} v_i f \, d\Omega$

NOTE: If L is self-adjoint then [S] in two methods is same.

① POINT COLLECTION METHOD

point matching method : $w_i = \delta = \begin{cases} \infty & \text{at pt } i \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow R_i = [L\{v\}^T \{c\} - f]_{\text{at pt } i} = 0$$

no. of matching pts ≡ no. of unknown.

② Subdomain Collection Method

weighting functions ≡ constant over specific subdomain and 0 elsewhere

$$\Rightarrow R_i = \int_{\underset{\underset{i^{th}\text{ subdomain}}{\uparrow}}{\Omega_i}} (\mathcal{L}\{v\}^T\{c\} - f)\, d\Omega = 0$$

③ Least squares method

minimize absolute square residual error

$$I = \frac{1}{2}\int_{\Omega} r^2\, d\Omega$$

ie. $$\frac{\partial I}{\partial c_i} = \int_{\Omega} \mathcal{L} v_i \, (\mathcal{L}\{v\}^T\{c\} - f)\, d\Omega = 0$$

$\boxed{\text{example}}$: $$\frac{\partial^2 \phi}{\partial x^2} = x+1$$

$$\Rightarrow R_i = \int_0^1 w_i \left(\frac{\partial^2 \tilde{\Phi}}{\partial x^2} - x - 1\right) dx = 0$$

$$\tilde{\Phi} = x + c_3(x^2 - x) + c_4(x^3 - x) \Rightarrow \frac{\partial^2 \tilde{\Phi}}{\partial x^2} = 2c_3 + 6c_4 x$$

$$= x + c_3 w_1 + c_4 w_2$$

$$\Rightarrow R_1 = \int_0^1 (x^2 - x)\,((6c_4 - 1)x + (2c_3 - 1))\, dx = 0$$

$$\Rightarrow \frac{1}{3}c_3 + \frac{1}{2}c_4 - \frac{1}{4} = 0$$

and

$$R_2 = \int_0^1 (x^3 - x)\,((6c_4 - 1)x + (2c_3 - 1))\, dx = 0$$

$$\Rightarrow \frac{1}{2}c_3 + \frac{4}{5}c_4 - \frac{23}{60} = 0$$

On solving: $c_3 = \frac{1}{2}$, $c_4 = \frac{1}{6}$

NOTE: Important step in Ritz and Glaerkin method — selection of trial function defined over entire solution domain. that can approx. represent true solution.

★ Finite Element Method (FEM)

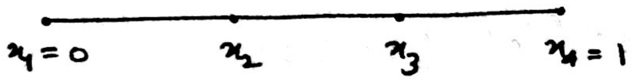
entire domain divided into small subdomain

↓

trial functions defined for each subdomain

example

$$\begin{cases} \dfrac{\partial^2 \phi}{\partial x^2} = x+1 , \quad 0<x<1 \\ \phi|_{x=0} = 0 , \quad \phi|_{x=1} = 1 \end{cases}$$



Let divide domain to subdomains arbitrary.

$(x_1, x_2), (x_2, x_3), (x_3, x_4)$

Now, chose $x_2 = \frac{1}{3}$, $x_3 = \frac{2}{3}$

{for simplicity}

Assume linear variation of $\phi(x)$ over each subdomain:

$$\tilde{\phi}(x) = \phi_i \frac{(x_{i+1} - x)}{(x_{i+1} - x_i)} + \phi_{i+1} \frac{(x - x_i)}{(x_{i+1} - x_i)} , \quad x_i \leq x \leq x_{i+1}$$

unknown constants

Apply b.c. $\tilde{\phi}(x)$ for $\underset{=0}{x_1} < x < x_2$: $\tilde{\phi}(x_1) = \phi_i = 0$

for $x_3 < x < \underset{=1}{x_4}$, $\tilde{\phi}(x_4) = \phi_4 = 1$

$\phi_2 , \phi_3 = ??$

①

Apply Ritz method

RITZ FEM

$$F = \frac{1}{2} \int_0^1 \left(\frac{d\tilde{\phi}}{dx} \right)^2 dx + \int_0^1 (x+1) \tilde{\phi} \, dx$$

$$= \sum_{i=1}^{3} \left[\frac{1}{2} \int_{x_i}^{x_{i+1}} \left(\frac{\phi_{i+1} - \phi_i}{x_{i+1} - x_i} \right)^2 dx + \int_{x_i}^{x_{i+1}} (x+1) \left(\phi_i \frac{(x_{i+1} - x)}{(x_{i+1} - x_i)} + \phi_{i+1} \left(\frac{x - x_i}{x_{i+1} - x_i} \right) \right) d \right.$$

$$= \sum_{i=1}^{3} \frac{1}{2} (x_{i+1} - x_i) \left[\left(\frac{\phi_{i+1} - \phi_i}{x_{i+1} - x_i} \right)^2 + \phi_{i+1} \left(\frac{2}{3} x_{i+1} + \frac{1}{3} x_i + 1 \right) \right.$$

$$\left. + \phi_i \left(\frac{2}{3} x_i + \frac{1}{3} x_{i+1} + 1 \right) \right]$$

Further simplify

$$F = 3\phi_2^2 + 3\phi_3^2 - 3\phi_2\phi_3 + \frac{4}{9}\phi_2 - \frac{22}{9}\phi_3 + \frac{49}{27}$$

$x_2 = 1/3$
$x_3 = 2/3$
$x_4 = 1$
$\phi_1 = 0$
$\phi_4 = 1$

Minimize F :

$$\frac{\partial F}{\partial \phi_2} = 0 \quad , \quad \frac{\partial F}{\partial \phi_3} = 0 \quad \Rightarrow \phi_2 = \frac{14}{81}$$

$$\phi_3 = \frac{40}{81}$$

② Apply Galerkin's method

$$w_i = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & , x_{i-1} < x < x_i \\[2ex] \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & , x_i < x < x_{i+1} \end{cases}$$

for $i = 2, 3$

NOTE: As long as trial function form a complete basis for the problem, one always obtains the exact sol^n even if diff. weighted residual methods are used.

GALERKIN FEM

Observe $\tilde{\Phi}$ can be differenciated only once.

$$\int_{x_{i-1}}^{x_{i+1}} w_i \frac{\partial^2 \tilde{\Phi}}{\partial x^2} dx = \cancelto{0}{w_i \frac{\partial \tilde{\Phi}}{\partial x}\Bigg|_{x_{i-1}}^{x_{i+1}}} - \int_{x_{i-1}}^{x_{i+1}} \frac{dw_i}{dx} \frac{d\tilde{\Phi}}{dx} dx$$

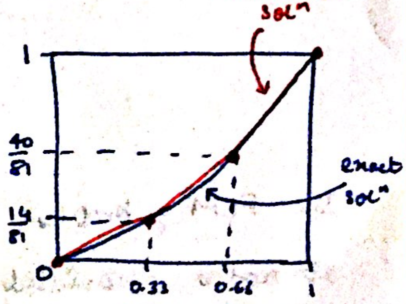
$$\Rightarrow R_i = \int_{x_{i-1}}^{x_{i+1}} w_i (L\tilde{\Phi} - f) dx = -\int_{x_{i-1}}^{x_{i+1}} \frac{dw_i}{dx} \frac{d\tilde{\Phi}}{dx} dx - \int_{x_{i-1}}^{x_{i+1}} w_i (x+1) dx$$

For $R_i = 0$

$$\Rightarrow \int_{x_{i-1}}^{x_{i+1}} \frac{dw_i}{dx} \frac{d\tilde{\Phi}}{dx} dx + \int_{x_{i-1}}^{x_{i+1}} w_i (x+1) dx = 0$$

Substitute w_i, $\tilde{\Phi}$ for $i = 2, 3$ and get

$$\phi_2 = \frac{14}{81} \quad , \quad \phi_3 = \frac{40}{81}$$



NOTE: * As number of subdivisions increased, more accurate is approx. solution.

- In FEM, as subdomains are small, basic functions defined over subdomain can be quite simple.
- In FEM, trial function is combination of a set of basic functions defined over subdomains.

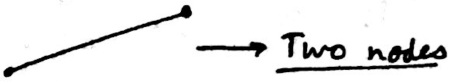
✶ Basic Steps of FEM

1. Discretization / sub-division of subdomain
2. Selection of interpolation functions
3. Formulation of system of equations (using Ritz or Galer method)
4. Solutions of system of equations

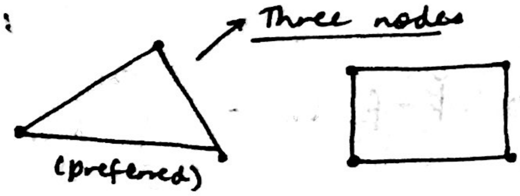
I. Domain discretization

$$\underset{\substack{\uparrow \\ \text{domain}}}{\Omega} \longrightarrow \underset{\substack{\uparrow \\ \text{subdomains} \\ \text{or} \\ \text{elements}}}{\Omega^e} \quad , \quad e = 1, 2, 3, \ldots\ldots, M$$

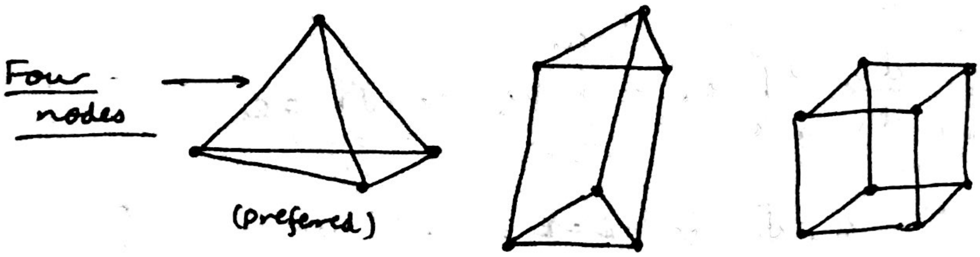
✶ One - dimensional domain :



✶ Two - dimensional domain :



✶ Three - dimensional domain :



In FEM, problem is formulated in terms of unknown functions ϕ at nodes associated with elements.

↓

It is necessary to describe these nodes.

- coordinate values
- local number ← indicates position in element
- global number ← indicates position in entire system

NOTE: numbering of nodes & elements require some stratergy as then computer storage and processing cost can be reduced.

(BANDWIDTH MATRIX SOLN METHOD)

II. Selection of interpolation function

STEP-I: └ Polynomial
- → linear — widely used for its simplicity
- → quadratic
- → cubic

STEP II: Unknown sol^n in element e:

$$\tilde{\phi}^e = \sum_{j=1}^{\overset{\text{no. of nodes}}{n}} N_j^e \, \phi_j^e = \{N^e\}^T \{\phi^e\} = \{\phi^e\}^T \{N^e\}$$

interpolation function for node j. Also called expansion/basis function.

value of ϕ at node j of element.

NOTE: * Highest order of N_j^e = order of element

example: if N_j^e is linear ⇒ e is linear element

* N_j^e is zero outside the element.

III. Formulation of system of Equations

Consider
$$\begin{cases} L\phi = f & , \text{in } \Omega \\ \text{Boundary cond}^n \text{ on boundary } \Gamma \text{ that encloses } \Omega \end{cases}$$

(A) Via Ritz method

Functional, $F(\tilde{\phi}) = \sum_{e=1}^{\overset{\text{total no. of elements}}{M}} F^e(\tilde{\phi}^e)$

where
$$F^e(\tilde{\phi}^e) = \frac{1}{2}\langle \tilde{\phi}^e, L\tilde{\phi}^e \rangle - \frac{1}{2}\langle f, \tilde{\phi}^e \rangle - \frac{1}{2}\langle \tilde{\phi}^e, f \rangle$$

$$= \frac{1}{2}\int_{\Omega^e} \tilde{\phi}^e L \tilde{\phi}^e \, d\Omega - \int_{\Omega^e} f \tilde{\phi}^e \, d\Omega$$

$$= \frac{1}{2}\{\phi^e\}^T \int_{\Omega^e} \{N^e\} L \{N^e\}^T d\Omega \, \{\phi^e\}$$

$$- \{\phi^e\}^T \int_{\Omega^e} f \{N^e\} \, d\Omega$$

$$\Rightarrow \quad F^e = \frac{1}{2}\{\phi^e\}^T[K^e]\{\phi^e\} - \{\phi^e\}^T\{b^e\}$$

where $K^e_{ij} = \int_{\Omega^e} N^e_i \, \mathcal{L} N^e_j \, d\Omega$

symmetric as $\mathcal{L}$ is self-adjoint

$$b^e_i = \int_{\Omega^e} f N^e_i \, d\Omega$$

$$\Rightarrow \quad F = \sum_{e=1}^{M} F^e = \sum_{e=1}^{M}\left(\frac{1}{2}\{\phi^e\}^T[K^e]\{\phi^e\} - \{\phi^e\}^T\{b^e\}\right)$$

$$= \frac{1}{2}\{\phi\}^T[K]\{\phi\} - \{\phi\}^T\{b\}$$ — adopting global numbers

$[K]$: N×N symm. matrix

$\{\phi\}$: N×1 unknown vector of expansion coeff.

$\{b\}$: N×1 known vector

where $N \equiv$ total no. of unknowns/nodes

Obtain system of equations by imposing $\delta F = 0$.

or $\dfrac{\partial F}{\partial \phi_i} = 0$, $\quad i = 1, 2, 3, \ldots, N$

$$\Rightarrow \quad \frac{1}{2}\sum_{j=1}^{N}(K_{ij} + K_{ji})\,\phi_j - b_i = 0$$

$$\Rightarrow \quad \sum_{j=1}^{N} K_{ij}\,\phi_j - b_i = 0 \qquad [\because K_{ij} = K_{ji}]$$

$$\Rightarrow \quad \boxed{[K]\{\phi\} = \{b\}}$$

(B) via Galerkin's method

$$R^e_i = \int_{\Omega_e} N^e_i\,(\mathcal{L}\tilde{\phi}^e - f)\,d\Omega \quad , i = 1, 2, 3, \ldots, n$$

$$= \int_{\Omega_e} N^e_i\,\mathcal{L}\{N^e\}^T d\Omega\,\{\phi^e\} - \int_{\Omega^e} f N^e_i\,d\Omega \quad , i = 1, 2, 3, \ldots$$

$$\Rightarrow \quad \{R^e\} = [K^e]\{\phi^e\} - \{b^e\}$$

$[K^e]$: not necessary symmetric

Since expansion, and therefore, the weighing function associated with a node, spans all elements directly connected to node, the weighted residual, $R_i \equiv$ summation over elements directly connected to node i.

$$\Rightarrow \{R\} = \sum_{e=1}^{M} \{\bar{R}^e\}$$

$$= \sum_{e=1}^{M} \left([\bar{K}^e]\{\bar{\phi}^e\} - \{\bar{b}^e\}\right)$$

NOTE: Overbar denotes vector has been expanded or augmented (by zero filling)

System of equations : $\{R\} = \{0\}$

$$\Rightarrow \sum_{e=1}^{M} \left([\bar{K}^e]\{\bar{\phi}^e\} - \{\bar{b}^e\}\right) = \{0\}$$

$$\Rightarrow \boxed{[K]\{\phi\} = \{b\}}$$

* TYPES OF BOUNDARY CONDN

- → Derichlet b.c. ← imposed explicitly
- → Neumann b.c.
  - * requires normal derivative of ϕ to vanish at boundary
  - * imposed implicitly ie. natural b.c.

* STEPS INVOLVED IN FORMULATION

STEP 1 : form elemental equation using F/G method

ie. $F^e = \frac{1}{2}\{\phi^e\}^T[K^e]\{\phi^e\} - \{\phi^e\}^T\{b^e\}$

or $\{R^e\} = [K^e]\{\phi^e\} - \{b^e\}$

STEP 2 : Summation over all elements to form system of eqn — $\boxed{\text{ASSEMBLY}}$

STEP 3 : Impose boundary conditions to obtain final form of system of eqn.

1. **Deterministic type**

$$[K]\{\phi\} = \{b\}$$

↑ results from inhomogenous DE / B.C. or both

2. **Eigenvalue type**

$$[A]\{\phi\} = \lambda [B]\{\phi\}$$

↑ results from homogenous DE and B.C.

* After solving for $\{\phi\}$, we can post-process data to make plots, find out desired quantities etc.

★ Alternate presentation of FEM formulation

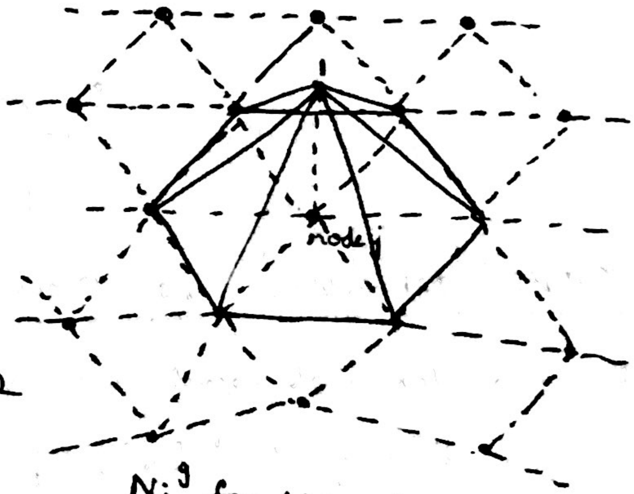
$$\tilde{\phi} = \sum_{e=1}^{M}\sum_{j=1}^{n} N_j^e \phi_j^e$$

$$\Rightarrow \tilde{\phi} = \sum_{j=1}^{N} \underset{\substack{\uparrow \\ \Rightarrow \text{global number}}}{N_j^g} \phi_j$$

Now, functional

$$F(\tilde{\phi}) = \frac{1}{2}\int_\Omega \tilde{\phi} L \tilde{\phi}\, d\Omega - \int_\Omega f \tilde{\phi}\, d\Omega$$

$$\Rightarrow \frac{\partial F}{\partial \phi_i} = \sum_{j=1}^{N} \phi_j \int N_i^g L N_j^g\, d\Omega - \int_\Omega f N_i^g\, d\Omega$$



$\underset{\substack{\uparrow \\ \text{nonzero only within elements connected to node j}}}{N_j^g}$ for linear triangular ele.

Imposing stationary reqmt. $\frac{\partial F}{\partial \phi_i} = 0$, $i = 1, 2, \ldots, N$

$$\Rightarrow [K]\{\phi\} = \{b\}$$

where

$$K_{ij} = \int_\Omega N_i^g L N_j^g\, d\Omega$$

$$b_i = \int_\Omega f N_i^g\, d\Omega$$

However, explicit expression N_j^g is not readily available. In this formulation, relation b/w local & global numbers is used to find N_j^g for given node no. while earlier it was used in process of assembly.

One-Dimensional F

Consider B.V.P.

$$\begin{cases} -\dfrac{d}{dx}\left(\alpha \dfrac{d\phi}{dx}\right) + \beta\phi = f \quad , \text{ in } (0,L) \\ \phi\Big|_{x=0} = p \\ \left[\alpha \dfrac{d\phi}{dx} + \gamma\phi\right]_{x=L} = q \end{cases}$$

Derichlet condn $\rightarrow \phi|_{x=0} = p$

Neumann condn when $\gamma = 0$ $\rightarrow \left[\alpha \frac{d\phi}{dx} + \gamma\phi\right]_{x=L} = q$

NOTE: If α has discontinuity say at x_d then so then function ϕ should satisfy continuity condn:

$$\phi\Big|_{x=x_d^+} = \phi\Big|_{x=x_d^-}$$

and

$$\left[\alpha \frac{d\phi}{dx}\right]_{x=x_d^+} = \left[\alpha \frac{d\phi}{dx}\right]_{x=x_d^-}$$

* Equivalent variational problem

$$\begin{cases} \delta F(\phi) = 0 \\ \phi|_{x=0} = p \end{cases}$$

where,

$$\underset{\uparrow}{F}(\phi) = \frac{1}{2}\int_0^L \left[\alpha\left(\frac{d\phi}{dx}\right)^2 + \beta\phi^2\right]dx - \int_0^L f\phi\, dx + \left[\frac{\gamma}{2}\phi^2 - q\phi\right]_{x=L}$$

Valid for both real and complex α, β and γ.

Proof:

First variation of $F(\phi)$, $\delta F = \lim_{e\to 0} \dfrac{F(\phi + e\,\delta F) - F(\phi)}{e}$

$$\Rightarrow \delta F = \lim_{e\to 0} \frac{1}{e}\left[\frac{1}{2}\int_0^L \left[\alpha e^2\left(\frac{d\delta\phi}{dx}\right)^2 + 2\alpha e \frac{d\phi}{dx}\frac{d\delta\phi}{dx} + \beta((\delta\phi)^2 e^2 + 2e\phi\delta\phi)\right]dx - \int_0^L f e\,\delta\phi\, dx + \left[\frac{\gamma}{2}((\delta\phi)^2 e^2 + 2e\phi\,\delta\phi) - qe\,\delta\phi\right]\right]_{x=L}$$

$$\Rightarrow \delta F = \int_0^L \left[\alpha \frac{d\phi}{dx}\frac{d\delta\phi}{dx} + \beta\phi\,\delta\phi\right]dx + [\gamma(\phi - q)\delta\phi]_{x=L} - \int_0^L f\,\delta F\, dx$$

I. Assume α is continuous in Ω

$$\delta F(\phi) = \int_0^L \left[-\frac{d}{dx}\left(\alpha \frac{d\phi}{dx}\right) + \beta\phi\right]\delta\phi\, dx + \left[\alpha \frac{d\phi}{dx}\, \delta\phi\right]_{x=0}^{x=L} \left\{ \begin{array}{l} \delta\phi|_{x=0} = 0 \\ \because \phi|_{x=0} \equiv \text{const} \end{array} \right\}$$

$$+ \left[(\gamma\phi - q)\,\delta\phi\right]_{x=L} - \int_0^L f\,\delta\phi\, dx$$

$$= \int_0^L \left[-\frac{d}{dx}\left(\alpha \frac{d\phi}{dx}\right) + \beta\phi\right]\delta\phi\, dx + \left[\left(\alpha\frac{d\phi}{dx} + \gamma\phi - q\right)\delta\phi\right]_{x=L}$$

$$- \int_0^L f\,\delta\phi\, dx$$

Imposing stationary reqmt $\delta F = 0$

$$\Rightarrow -\frac{d}{dx}\left(\alpha\frac{d\phi}{dx}\right) + \beta\phi - f = 0$$ ← Euler equation of defined $F(\phi)$.

and

$$\left(\alpha\frac{d\phi}{dx} + \gamma\phi - q\right)_{x=L} = 0$$ ← natural b.c. as its satisfied automatically in process of min/max of functional.

II. Suppose α is discontinuous at x_d

We can't apply integration by parts over entire Ω directly. Instead we integrate over subdomains : $(0, x_d)$ and (x_d, L)

$$\delta F' = \delta F + \left[\alpha\frac{d\phi}{dx}\,\delta\phi\right]_{x=x_d^+} - \left[\alpha\frac{d\phi}{dx}\,\delta\phi\right]_{x=x_d^-}$$

As ϕ is continuous at x_d, so is $\delta\phi$

Apply stationary reqmt to get additional condn:

$$\left[\alpha\frac{d\phi}{dx}\right]_{x=x_d^+} = \left[\alpha\frac{d\phi}{dx}\right]_{x=x_d^-}$$ ← another natural condn

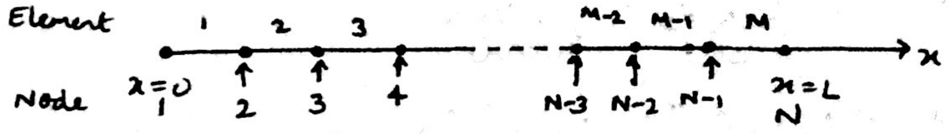
NOTE: If α, β, γ are purely real, another functional possible is

$$F(\phi) = \frac{1}{2}\int_0^L \left[\alpha\left|\frac{d\phi}{dx}\right|^2 + \beta|\phi|^2\right] dx + \frac{1}{2}\left[\gamma|\phi|^2 - q^*\phi - q\phi^*\right]_{x=L}$$

$$-\frac{1}{2}\int_0^L (f^*\phi + f\phi^*)\, dx$$

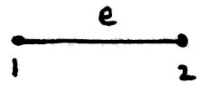
★ Finite element analysis

1. Discretization



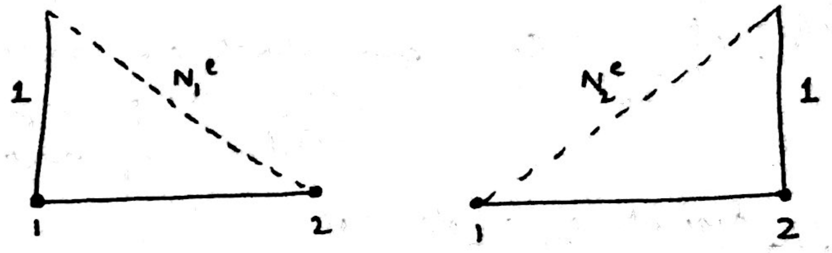
- local and global systems are related by

$$\boxed{x_1^e = x_e \quad \text{and} \quad x_2^e = x_{e+1}} \quad , \text{ for } e = 1, 2, 3, \ldots, M$$



2. Interpolation

apply linear functions for interpolation : $\phi^e(x) = a^e + b^e x$



basis functions : $N_1^e(x) = \dfrac{x_2^e - x}{l^e}$ $\qquad$ $N_2^e(x) = \dfrac{x - x_1^e}{l^e}$

where $l^e \equiv x_2^e - x_1^e$

Also $N_j^e(x_i^e) = \delta_{ij} = \begin{cases} 1 , & i = j \\ 0 , & i \neq j \end{cases}$

At two nodes
$$\phi_1^e = a^e + b^e x_1^e$$
$$\phi_2^e = a^e + b^e x_2^e$$

$$\Rightarrow b^e = \frac{\phi_2^e - \phi_1^e}{x_2^e - x_1^e} , \quad a^e = \frac{\phi_2^e x_1^e - \phi_1^e x_2^e}{x_1^e - x_2^e}$$

$$\Rightarrow \phi^e(x) = \phi_2^e \left(\frac{x - x_1^e}{l^e} \right) + \phi_1^e \left(\frac{x_2^e - x}{l^e} \right)$$
$$= \phi_2^e N_2^e + \phi_1^e N_1^e$$

$$\Rightarrow \boxed{\phi^e(x) = \sum_{j=1}^{2} N_j^e(x) \, \phi_j^e}$$

3. Formulation via:

A : Ritz method

STEP1 $$F^e(\phi^e) = \frac{1}{2}\int_{x_1^e}^{x_2^e}\left[\alpha\left(\frac{d\phi^e}{dx}\right)^2 + \beta(\phi^e)^2\right]dx - \int_{x_1^e}^{x_2^e}\phi^e f\,dx$$

and

$$F(\phi) = \sum_{e=1}^{M} F^e(\phi^e).$$

[Take $\gamma = q = 0$]
i.e. homogenous Neuman condn

Now,

$$F^e(\phi^e) = \sum_{j=1}^{2}\left\{\frac{1}{2}\int_{x_1^e}^{x_2^e}\left[\alpha\left(\frac{d}{dx}(\phi_j^e N_j^e)\right)^2 + \beta(\phi_j^e N_j^e)^2\right]dx - \int_{x_1^e}^{x_2^e}\phi_j^e N_j^e f\,dx\right\}$$
$$+ \int_{x_1^e}^{x_2^e}\left[\alpha\frac{dN_1^e}{dx}\frac{dN_2^e}{dx}dx\,\phi_1^e\phi_2^e + \beta\,\phi_1^e\phi_2^e N_1^e N_2^e\,dx\right]$$

$$\Rightarrow \frac{\partial F^e}{\partial \phi_i^e} = \sum_{j=1}^{2}\phi_j^e\int_{x_1^e}^{x_2^e}\left(\alpha\frac{dN_i^e}{dx}\frac{dN_j^e}{dx} + \beta N_i^e N_j^e\right)dx - \int_{x_1^e}^{x_2^e}N_i^e f\,dx$$

$$\Rightarrow \boxed{\left\{\frac{\partial F^e}{\partial \phi_i^e}\right\} = [K^e]\{\phi^e\} - \{b^e\}}$$

Formation of elemental eqn

where $\phi^e = (\phi_1^e, \phi_2^e)'$

$$K_{ij}^e = \int_{x_1^e}^{x_2^e}\left(\alpha\frac{dN_i^e}{dx}\frac{dN_j^e}{dx} + \beta N_i^e N_j^e\right)dx$$

$$b_i^e = \int_{x_1^e}^{x_2^e} N_i^e f\,dx$$

If α, β are constants / approximated within each element then

$$K^e = \begin{bmatrix} \frac{\alpha^e}{l^e} + \beta^e\frac{l^e}{3} & -\frac{\alpha^e}{l^e} + \beta^e\frac{l^e}{6} \\ -\frac{\alpha^e}{l^e} + \beta^e\frac{l^e}{6} & \frac{\alpha^e}{l^e} + \beta^e\frac{l^e}{3} \end{bmatrix}$$

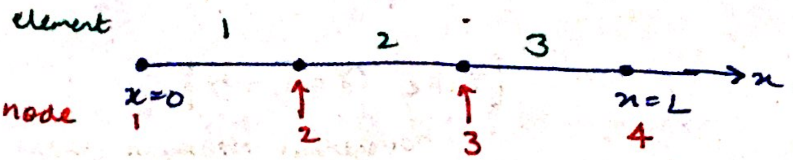
$$b_1^e = b_2^e = f^e\frac{l^e}{2}$$

where $\alpha^e, \beta^e, f^e \equiv$ values within e^{th} element

STEP 2

$$\left\{\frac{\partial F}{\partial \phi}\right\} = \sum_{e=1}^{M} \left\{\overline{\frac{\partial F^e}{\partial \phi^e}}\right\} = \sum_{e=1}^{M}\left([\bar{K}^e]\{\bar{\phi}^e\} - \{\bar{b}^e\}\right) = \{0\}$$

For illustration conside $M=3, N=4$



$$[\bar{K}^{(1)}] = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \{\bar{\phi}^{(1)}\} = \begin{Bmatrix} \phi_1^{(1)} \\ \phi_2^{(1)} \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow [\bar{K}^{(1)}]\{\bar{\phi}^{(1)}\} = \begin{Bmatrix} K_{11}^{(1)}\phi_1^{(1)} + K_{12}^{(1)}\phi_2^{(1)} \\ K_{21}^{(1)}\phi_1^{(1)} + K_{22}^{(1)}\phi_2^{(1)} \\ 0 \\ 0 \end{Bmatrix}$$

Similarly we can find out $[\bar{K}^{(i)}][\bar{\phi}^{(i)}]$ for $i=2,3$

$$\Rightarrow \sum_{i=1}^{3} [\bar{K}^{(i)}][\bar{\phi}^{(i)}] = \begin{Bmatrix} K_{11}^{(1)}\phi_1^{(1)} + K_{12}^{(1)}\phi_2^{(1)} \\ K_{21}^{(1)}\phi_1^{(1)} + K_{22}^{(1)}\phi_2^{(1)} + K_{11}^{(2)}\phi_1^{(2)} + K_{12}^{(2)}\phi_2^{(2)} \\ K_{21}^{(2)}\phi_1^{(2)} + K_{22}^{(2)}\phi_2^{(2)} + K_{11}^{(3)}\phi_1^{(3)} + K_{12}^{(3)}\phi_2^{(3)} \\ K_{21}^{(3)}\phi_1^{(3)} + K_{22}^{(3)}\phi_2^{(3)} \end{Bmatrix}$$

$$= \underbrace{\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} + K_{11}^{(2)} & K_{12}^{(2)} & 0 \\ 0 & K_{21}^{(2)} & K_{22}^{(2)} + K_{11}^{(3)} & K_{12}^{(3)} \\ 0 & 0 & K_{21}^{(3)} & K_{22}^{(3)} \end{bmatrix}}_{[K]} \underbrace{\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix}}_{\{\phi\}}$$

using relation b/w local and globe node number

and

$$\sum_{e=1}^{3} \{\bar{b}^e\} = \underbrace{\begin{Bmatrix} b_1^{(1)} \\ b_2^{(1)} + b_1^{(2)} \\ b_2^{(2)} + b_1^{(3)} \\ b_2^{(3)} \end{Bmatrix}}_{\{b\}}$$

Assembly

$$\Rightarrow \boxed{[K]\{\phi\} = \{b\}}$$

↳ To make procedure simpler : $\sum_{e=1}^{M} [\bar{K}^e]\{\bar{\phi}^e\} = \sum_{e=1}^{M} [\bar{K}^e]\{\phi\}$

Observation

Replace $\{\bar{\phi}^e\} \longrightarrow \{\phi\}$

$$\Rightarrow [K] = \sum_{e=1}^{M} [\bar{K}^e]$$

ie. $K_{11} = K_{11}^{(1)}$, $K_{NN} = K_{22}^{(M)}$

$$K_{ii} = K_{22}^{(i-1)} + K_{11}^{(i)}$$

$$K_{i+1,i} = K_{i,i+1} = K_{12}^{(i)}$$

NOTE: Suppose we consider general form of b.c. (of third kind) at $x=L$:

extra term → $\underset{\substack{\uparrow \\ b\,=\,\text{boundary}}}{F_b}(\phi) = \left[\frac{\gamma}{2}\phi^2 - q\phi\right]_{x=L}$

F_b only contains ϕ_N ⇒ $\frac{\partial F_b}{\partial \phi_N} = \gamma\phi_N - q$

$$\Rightarrow K_{NN} = \frac{\alpha^{(M)}}{L^{(M)}} + \beta^{(M)}\frac{L^{(M)}}{3} + \gamma$$

and

$$b_N = f^{(M)}\frac{L^{(M)}}{2} + q$$

STEP-3 B.C. of third kind already imposed.

$\phi|_{x=0} = p$ ← Derichlet condn

Imposition of boundary conditions

ie. $\phi_1 = p$

⇔ $K_{11} = 1$, $b_1 = p$, $K_{1j} = 0$, $j = 2, 3, \ldots, N$

ie.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ K_{21} & K_{22} & \cdots & \cdots \\ \vdots & & & \\ \vdots & \cdots & \cdots & K_{44} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} p \\ b_2 \\ b_3 \\ b_4 \end{Bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & K_{22} & K_{23} & K_{24} \\ 0 & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & K_{44} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} p \\ b_2 - pK_{21} \\ b_3 - pK_{31} \\ b_4 - pK_{41} \end{Bmatrix}$$

$$\Rightarrow \begin{bmatrix} K_{22} & K_{23} & K_{24} \\ \vdots & \vdots & \vdots \\ K_{42} & \cdots & K_{44} \end{bmatrix} \begin{Bmatrix} \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} b_2 - K_{21}\, p \\ b_3 - K_{31}\, p \\ b_4 - K_{41}\, p \end{Bmatrix}$$

No. of eqⁿ reduced

B : via Galerkin's Method

$$r = -\frac{d}{dx}\left(\alpha \frac{d\phi}{dx}\right) + \beta\phi - f$$

STEP 1

Weighted residual integral for eth element

$$R_i^e = \int_{x_1^e}^{x_2^e} N_i^e\, r \, dx \qquad ; \; i = 1,2$$

$$= \int_{x_1^e}^{x_2^e} N_i^e \left[-\frac{d}{dx}\left(\alpha \frac{d\phi}{dx}\right) + \beta\phi \right] dx - \int_{x_1^e}^{x_2^e} N_i^e f \, dx$$

$$= \int_{x_1^e}^{x_2^e} \left(\alpha \frac{dN_i^e}{dx}\frac{d\phi}{dx} + \beta N_i^e \phi \right) dx - \int_{x_1^e}^{x_2^e} N_i^e f\, dx - \alpha N_i^e \frac{d\phi}{dx}\Bigg|_{x_1^e}^{x_2^e}$$

$$= \sum_{j=1}^{2} \phi_j^e \int_{x_1^e}^{x_2^e} \left(\alpha \frac{dN_i^e}{dx}\frac{dN_j^e}{dx} + \beta N_i^e N_j^e \right) dx - \int_{x_1^e}^{x_2^e} N_i^e f \, dx$$

$$- \alpha N_i^e \frac{d\phi}{dx}\Bigg|_{x_1^e}^{x_2^e}$$

$$\Rightarrow \{R^e\} = [K^e]\{\phi^e\} - \{b^e\} - \{g^e\}$$

where

$$K_{ij} = \int_{x_1^e}^{x_2^e} \left(\alpha \frac{dN_i^e}{dx}\frac{dN_j^e}{dx} + \beta N_i^e N_j^e \right) dx$$

$$b_i = \int_{x_1^e}^{x_2^e} N_i^e f \, dx$$

$$g_i = \alpha N_i^e \frac{d\phi}{dx}\Bigg|_{x_1^e}^{x_2^e} = \mp \alpha \frac{d\phi}{dx}\Bigg|_{x = x_i^e}$$

STEP 2

$$\Rightarrow \{R\} = \sum_{e=1}^{M} \{\bar{R}^e\} = \sum_{e=1}^{M} \left([\bar{K}^e]\{\bar{\phi}^e\} - \{\bar{b}^e\} - \{\bar{g}^e\} \right) = \{0\}$$

$$\Rightarrow \boxed{[K]\{\phi\} = \{b\} + \{g\}}$$

Assembly of [K] and [b] same as that in Ritz method.

$$\{g\} = \sum_{e=1}^{3} \{\bar{g}^e\} = \begin{Bmatrix} g_1^{(1)} \\ g_2^{(1)} + g_1^{(2)} \\ g_2^{(2)} + g_1^{(3)} \\ g_2^{(3)} \end{Bmatrix}$$

Except g_1 and g_4, other elements of $\{g\}$:

$$g_i = g_2^{(i-1)} + g_1^{(i)} = \alpha \frac{d\phi}{dx}\Big|_{x=x_2^{(i-1)}} - \alpha \frac{d\phi}{dx}\Big|_{x=x_1^{(i)}}$$

$$= 0 \qquad \left[\because x_2^{(i-1)} = x_1^{(i)} = x_i\right]$$

$$\Rightarrow \{g\} = \begin{Bmatrix} -\alpha \frac{d\phi}{dx}\Big|_{x=x_1} \\ 0 \\ 0 \\ \alpha \frac{d\phi}{dx}\Big|_{x=x_4} \end{Bmatrix}$$

For a general problem with N nodes:

$$g_1 = -\alpha \frac{d\phi}{dx}\Big|_{x=x_1=0}$$

$$g_N = \alpha \frac{d\phi}{dx}\Big|_{x=x_N=L}$$

$$g_i = 0 \quad \text{for } i = 2, 3, \ldots, N-1$$

STEP-3

Apply b.c. to system of eqns; modifying system in same manner as in Ritz formation:

$b_1 = p$, $b_i \leftarrow b_i - K_{i1}\,p$; for $i = 2, 3, 4, \ldots$

$K_{11} = 1$, $K_{1i} = K_{i1} = 0$; for $i = 2, 3, 4, \ldots$

$x=0$: $$\underset{\substack{\uparrow \\ \text{Should be discarded ? How ?}}}{g_1} = -\alpha \frac{d}{dx}\left(\phi_1^e N_1^e + \phi_2^e N_2^e\right)\Big|_{x=0} = -\alpha\left(\phi_2^e - \underset{\substack{\uparrow \\ \phi_1 = p}}{\phi_1^e}\right)\Big|_{x=0}$$

$x=L$: $$g_N = \alpha \frac{d\phi}{dx}\Big|_{x=L} = \underset{\substack{\uparrow \\ \text{can be added to } b_N}}{q} - \underset{\substack{\uparrow \\ \text{can be added to } K_{NN}}}{\gamma\phi_N}$$

∴ System of equations same as that in Ritz formation.

4. Solution of System of Equations

System is symmetric tridiagonal matrix

↓

can be solved using Gaussian elimination

Let $\{a\} := a_i = K_{ii}\ ,\ i = 1, 2, \ldots, N-1, N$

$\{c\} := c_i = K_{i+1,i}\ ,\ i = 1, 2, 3, \ldots, N-1$

Reduce system into upper triangular form:

$$a_i \leftarrow a_i - \frac{c_{i-1}^2}{a_{i-1}}\ ,\ b_i \leftarrow b_i - \frac{c_{i-1}\, b_{i-1}}{a_{i-1}}$$

Verify this algorithm

for $i = 2, 3, 4, \ldots, N$

$$\Rightarrow \phi_N = \frac{b_N}{a_N}\ ,\ \phi_i = \frac{b_i - c_i\,\phi_{i+1}}{a_i} \quad \text{for } i = N-1, N-2, \ldots, 1$$

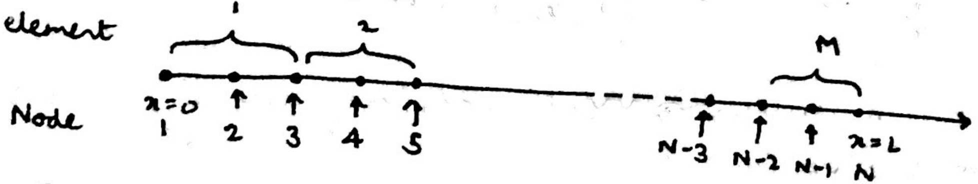
★ Higher Order Elements

ADVANTAGES → better results, higher accuracy without increasing number of nodes, cost-effective

DISADVANTAGES → (1) complexity of formulation

(2) increase in bandwidth of system of eq^{n}s

I. Quadratic elements



Each element has three nodes.

$$\phi^e(x) = a^e + b^e x + c^e x^2$$

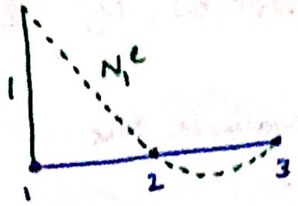
At the three nodes:

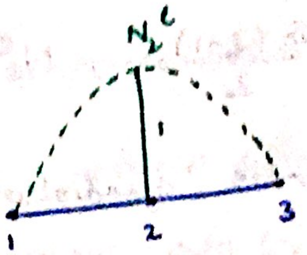
$$\phi_1^e = a^e + b^e x_1^e + c^e {x_1^e}^2$$

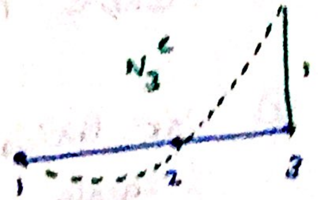
$$\phi_2^e = a^e + b^e x_2^e + c^e {x_2^e}^2$$

$$\phi_3^e = a^e + b^e x_3^e + c^e {x_3^e}^2$$

$$\Rightarrow \phi^e(x) = \sum_{j=1}^{3} N_j^e(x)\,\phi_j^e$$







$$N_1^e = \frac{(x-x_2^e)(x-x_3^e)}{(x_1^e-x_2^e)(x_1^e-x_3^e)} \qquad N_2^e = \frac{(x-x_1^e)(x-x_3^e)}{(x_2^e-x_1^e)(x_2^e-x_3^e)} \qquad N_3^e = \frac{(x-x_1^e)(x-x_2^e)}{(x_3^e-x_1^e)(x_3^e-x_2^e}$$

NOTE: $N_j^e(x_i^e) = \delta_{ij}$

Following similar procedure as for linear elements

$$[K^e]\{\phi^e\} = \{b^e\}$$

where $$K_{ij}^e = \int_{x_1^e}^{x_3^e} \left(\alpha \frac{dN_i^e}{dx}\frac{dN_j^e}{dx} + \beta N_i^e N_j^e\right) dx \ ; \ i,j = 1,2,3$$

$$b_i^e = \int_{x_1^e}^{x_3^e} N_i^e f \, dx \quad , \ i = 1,2,3$$

$\boxed{\text{CASE-1}}$: α, β, f are constants or can be approximated as constants within the element.

$$[K^e] = \begin{bmatrix} \alpha^e \frac{7}{3l^e} + \beta^e \frac{2l^e}{15} & -\alpha^e \frac{8}{3l^e} + \beta^e \frac{l^e}{15} & \alpha^e \frac{1}{3l^e} - \beta^e \frac{l^e}{30} \\ -\alpha^e \frac{8}{3l^e} + \beta^e \frac{l^e}{15} & \alpha^e \frac{16}{3l^e} + \beta^e \frac{8l^e}{15} & -\alpha^e \frac{8}{3l^e} + \beta^e \frac{l^e}{15} \\ \alpha^e \frac{1}{3l^e} - \beta^e \frac{l^e}{30} & -\alpha^e \frac{8}{3l^e} + \beta^e \frac{l^e}{15} & \alpha^e \frac{7}{3l^e} + \beta^e \frac{2l^e}{15} \end{bmatrix}$$

and

$$\{b^e\} = \left[\frac{l^e}{6} f^e, \ \frac{2l^e}{3} f^e, \ \frac{l^e}{6} f^e\right]^T$$

$\boxed{\text{CASE-2}}$: When α, β, γ cannot be approximated as constants

① Normalize the integral interval to $(-1,1)$: $\varepsilon = \frac{2(x-x_2^e)}{l^e}$

② $$K_{ij}^e = \int_{-1}^{1} \left(\frac{2}{l^e} \alpha \frac{dN_i^e}{d\varepsilon}\frac{dN_j^e}{d\varepsilon} + \frac{l^e}{2} \beta N_i^e N_j^e \right) d\varepsilon$$

$$b_i^e = \frac{l^e}{2} \int_{-1}^{1} N_i^e f \, d\varepsilon$$

integrals. procedure to evaluate the

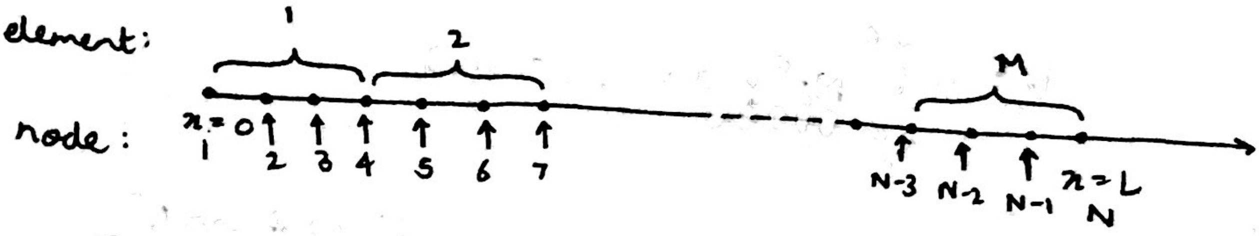
$$\int_{-1}^{1} P(\varepsilon)\, d\varepsilon = \sum_{i=1}^{n} W_i\, P(\varepsilon_i)$$

where $\varepsilon_i \equiv$ sampling points

$W_i \equiv$ associated weighing parameters

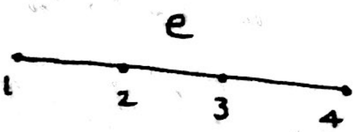
NOTE: In assembly procedure, bandwidth of system for this case is atleast 5.

II. Cubic Elements



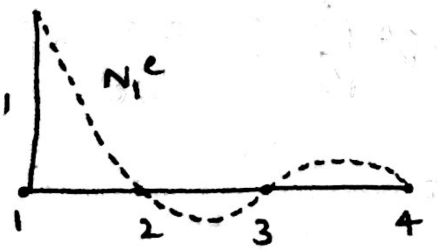
Each element has four nodes.

$$\phi^e(x) = a^e + b^e x + c^e x^2 + d^e x^3$$

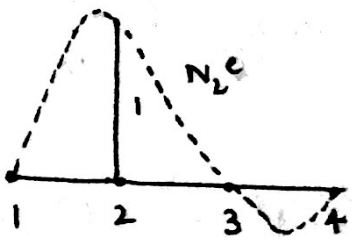


Similar to earlier, at the four nodes ϕ_j^e $(j = 1,2,3,4)$

$$\Rightarrow \phi^e(x) = \sum_{j=1}^{4} N_j^e(x)\, \phi_j^e$$

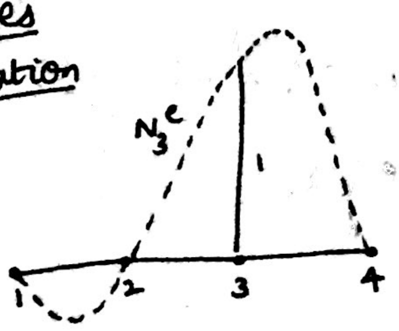


$$N_1^e = \frac{(x-x_2^e)(x-x_3^e)(x-x_4^e)}{(x_1^e-x_2^e)(x_1^e-x_3^e)(x_1^e-x_4^e)}$$

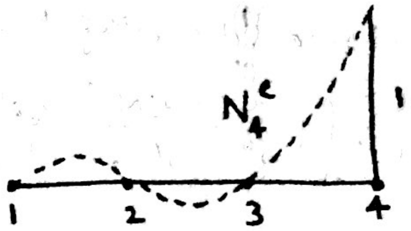


$$N_2^e = \frac{(x-x_1^e)(x-x_3^e)(x-x_4^e)}{(x_2^e-x_1^e)(x_2^e-x_3^e)(x_2^e-x_4^e)}$$

Lagrange's interpolation



$$N_3^e = \frac{(x-x_1^e)(x-x_2^e)(x-x_4^e)}{(x_3^e-x_1^e)(x_3^e-x_2^e)(x_3^e-x_4^e)}$$



$$N_4^e = \frac{(x-x_1^e)(x-x_2^e)(x-x_3^e)}{(x_4^e-x_1^e)(x_4^e-x_2^e)(x_4^e-x_3^e)}$$

CASE 1 : When α, β, f are constants for e^{th} element

CASE 2 : If α, β, f are not constants within e^{th} element

① Normalize integration interval into $(-1,1)$

$$\varepsilon = \frac{2(x - x_c^e)}{l^e} \quad \text{where } x_c^e \equiv \text{center of element}$$

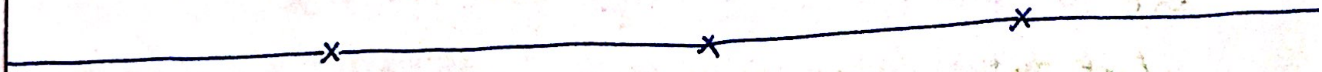
② $$N_1^e = -\frac{9}{16}\left(\varepsilon+\frac{1}{3}\right)\left(\varepsilon-\frac{1}{3}\right)(\varepsilon-1)$$

$$N_2^e = \frac{27}{16}(\varepsilon+1)\left(\varepsilon-\frac{1}{3}\right)(\varepsilon-1)$$

$$N_3^e = -\frac{27}{16}(\varepsilon+1)\left(\varepsilon+\frac{1}{3}\right)(\varepsilon-1)$$

$$N_4^e = \frac{9}{16}(\varepsilon+1)\left(\varepsilon+\frac{1}{3}\right)\left(\varepsilon-\frac{1}{3}\right)$$

③ Apply Gauss - legendre quadrature to evaluate integrals.



Consider B.V.P.

$$\begin{cases} -\frac{\partial}{\partial x}\left(\alpha_x \frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\alpha_y \frac{\partial \phi}{\partial y}\right) + \beta\phi = f & , (x,y) \in \Omega \\ \phi = p & \text{on } \Gamma_1 \\ \left(\alpha_x \frac{\partial \phi}{\partial x}\hat{x} + \alpha_y \frac{\partial \phi}{\partial y}\hat{y}\right)\cdot\hat{n} + \gamma\phi = q & \text{on } \Gamma_2 \end{cases}$$

where $\Gamma_1 + \Gamma_2 = \Gamma \equiv$ contour enclosing the area Ω

$\hat{n} \equiv$ outward normal unit vector

NOTE: If α_x and α_y have discontinuities and if there is no surface source of any kind at the discontinuity interface, then ϕ should satisfy continuity conditions

$$\phi^+ = \phi^- \text{ on } \Gamma_d$$

and

$$\left(\alpha_x^+ \frac{\partial \phi^+}{\partial x}\hat{x} + \alpha_y^+ \frac{\partial \phi^+}{\partial y}\hat{y}\right)\cdot\hat{n} = \left(\alpha_x^- \frac{\partial \phi^-}{\partial x}\hat{x} + \alpha_y^- \frac{\partial \phi^-}{\partial y}\hat{y}\right)\cdot\hat{n} \quad \text{on } \Gamma_d$$

where $\Gamma_d \equiv$ discontinuity interface.

* Equivalent variational formulation

$$\begin{cases} \delta F(\phi) = 0 \\ \phi = p \quad \text{on } \Gamma_1 \end{cases}$$

where
$$F(\phi) = \frac{1}{2}\iint_\Omega \left[\alpha_x\left(\frac{\partial \phi}{\partial x}\right)^2 + \alpha_y\left(\frac{\partial \phi}{\partial y}\right)^2 + \beta\phi^2\right] d\Omega + \int_{\Gamma_2}\left(\frac{\gamma}{2}\phi^2 - q\phi\right) d\Gamma - \iint_\Omega f\phi \, d\Omega$$

Proof:

I. Assume α_x and α_y are continuous over entire Ω.

$$\delta F(\phi) = \iint_\Omega \left[\alpha_x\left(\frac{\partial \phi}{\partial x}\right)\left(\frac{\partial \delta\phi}{\partial x}\right) + \alpha_y\left(\frac{\partial \phi}{\partial y}\right)\left(\frac{\partial \delta\phi}{\partial y}\right) + \beta\phi\,\delta\phi\right] d\Omega + \int_{\Gamma_2} (\gamma\phi - q)\,\delta\phi \, d\Gamma - \iint_\Omega f\,\delta\phi \, d\Omega$$

As $$\alpha_x \left(\frac{\partial\phi}{\partial x}\right)\left(\frac{\partial\, \delta\phi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\alpha_x \frac{\partial\phi}{\partial x}\delta\phi\right) - \left[\frac{\partial}{\partial x}\left(\alpha_x\frac{\partial\phi}{\partial x}\right)\right]\delta\phi$$

$$\alpha_y \left(\frac{\partial\phi}{\partial y}\right)\left(\frac{\partial\, \delta\phi}{\partial y}\right) = \frac{\partial}{\partial y}\left(\alpha_y \frac{\partial\phi}{\partial y}\delta\phi\right) - \left[\frac{\partial}{\partial y}\left(\alpha_y\frac{\partial\phi}{\partial y}\right)\right]\delta\phi$$

$$\iint_\Omega \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) d\Omega = \oint_\Gamma (U\hat{x}+V\hat{y})\cdot\hat{n}\, d\Gamma$$

$$\Rightarrow \delta F(\phi) = \iint_\Omega \left[-\frac{\partial}{\partial x}\left(\alpha_x\frac{\partial\phi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\alpha_y\frac{\partial\phi}{\partial y}\right) + \beta\phi - f\right]\delta\phi\, d\Omega$$

$$+ \oint_\Gamma \left(\alpha_x\frac{\partial\phi}{\partial x}\hat{x} + \alpha_y\frac{\partial\phi}{\partial y}\hat{y}\right)\cdot\hat{n}\;\delta\phi\, d\Gamma$$

$$+ \underbrace{\int_{\Gamma_2}(\gamma\phi - q)\,\delta\phi\, d\Gamma}$$

$$= \oint_\Gamma (\gamma\phi - q)\,\delta\phi\, d\Gamma \quad \left[\because \int_{\Gamma_1}(\gamma\phi-q)\,\delta\phi\, d\Gamma = 0 \text{ as } \begin{matrix}\phi = \text{const} \\ \text{over } \Gamma_1 \\ \Rightarrow \delta\phi = 0\end{matrix}\right]$$

$$= \iint_\Omega \left[-\frac{\partial}{\partial x}\left(\alpha_x\frac{\partial\phi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\alpha_y\frac{\partial\phi}{\partial y}\right) + \beta\phi - f\right]\delta\phi\, d\Omega$$

$$+ \int_{\Gamma_2}\left[\left(\alpha_x\frac{\partial\phi}{\partial x}\hat{x} + \alpha_y\frac{\partial\phi}{\partial y}\hat{y}\right)\cdot\hat{n} + \gamma\phi - q\right]\delta\phi\, d\Gamma$$

Imposing stationary reqmt.

$$-\frac{\partial}{\partial x}\left(\alpha_x\frac{\partial\phi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\alpha_y\frac{\partial\phi}{\partial y}\right) + \beta\phi - f = 0$$

and $$\left(\alpha_x\frac{\partial\phi}{\partial x}\hat{x} + \alpha_y\frac{\partial\phi}{\partial y}\hat{y}\right)\cdot\hat{n} + \gamma\phi - q = 0 \quad \text{on } \Gamma_2$$

II. Suppose α_x and α_y have abrupt changes, then we first split entire domain into several subdomains within which α_x and α_y are continuous.

$$\delta F' = \delta F + \int_{\Gamma_d}\left(\alpha_x^+\frac{\partial\phi^+}{\partial x}\hat{x} + \alpha_y^+\frac{\partial\phi^+}{\partial y}\hat{y}\right)\cdot\hat{n}^+\;\delta\phi^+\, d\Gamma$$

$$+ \int_{\Gamma_d}\left(\alpha_x^-\frac{\partial\phi^-}{\partial x}\hat{x} + \alpha_y^-\frac{\partial\phi^-}{\partial y}\hat{y}\right)\cdot\hat{n}^-\;\delta\phi^-\, d\Gamma$$

where $\hat{n}^+ = -\hat{n}^-$.

As ϕ is continuous across Γ_d , $\delta\phi^+ = \delta\phi^-$

Apply stationary reqmt to get another natural condn

$$\left(\alpha_x^+ \frac{\partial\phi^+}{\partial x}\hat{x} + \alpha_y^+ \frac{\partial\phi^+}{\partial y}\right)\cdot\hat{n}^+ + \left(\alpha_x^- \frac{\partial\phi^-}{\partial x}\hat{x} + \alpha_y^- \frac{\partial\phi^-}{\partial y}\hat{y}\right)\cdot\hat{n}^- = 0 \Big\}$$

continuity condn

NOTE: If $\alpha_x, \alpha_y, \beta, \gamma$ are real only then another functional which could be used is:

$$F(\phi) = \frac{1}{2}\iint_\Omega \left[\alpha_x\left|\frac{\partial\phi}{\partial x}\right|^2 + \alpha_y\left|\frac{\partial\phi}{\partial y}\right|^2 + \beta|\phi|^2\right]d\Omega$$
$$+\frac{1}{2}\int_{\Gamma_2}(\gamma|\phi|^2 - q^*\phi - q\phi^*)\,d\Gamma - \frac{1}{2}\iint_\Omega (f^*\phi + f\phi^*)\,d\Omega$$

★ Finite element analysis

1. Domain Discretization

We need to ensure:

i) no overlap or gaps b/w elements

ii) elements should be connected via their vertices
ie. no vertex of one element can be internal to the side of other

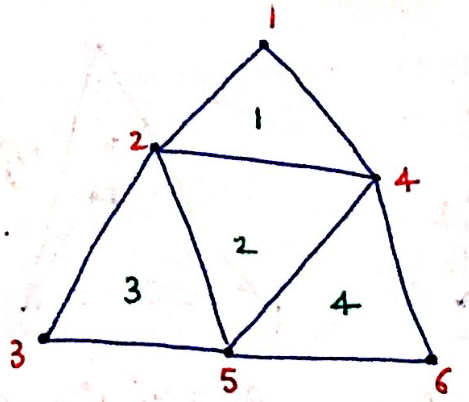
iii) avoid generating elements or elements having small inner angle as they tend to increase sol^n error

$$\uparrow$$
$$\left[\propto \frac{1}{\sin(\angle\text{smallest inner angle})}\right]$$

iv) all elements should be made close to equilateral

v) Although smaller elements result in more accurate sol^n, they produce more unknowns and increase computing time.

NOTE: Good practice : use small elements where sol^n is anticipated to have drastic variation, and larger elements in regions where variation is more gentle.



e	n(1,e)	n(2,e)	n(3,e)
1	2	4	1
2	5	4	2
3	3	5	2
4	5	6	4

connectivity array

:= n(i,e) where i = 1, 2, 3

e = 1, 2, ..., M

NOTE: Connectivity array contains global node no. indexed by local node no. i and element number e.

* similar array can be defined which relates to segments coincidents with Γ_2 with their nodes. ← Help to apply b.c.

* vector reqd to store global no. of nodes residing on Γ_1

↑ apply Derichlet b.c.

* Other data reqd for finite element formulation:

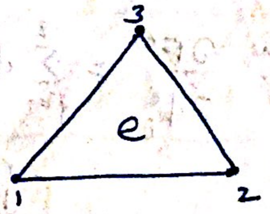
a) coordinates of nodes (x_i, y_i) for $i = 1, 2, \ldots, N$

b) values of $\alpha_x, \alpha_y, \beta, f$ for each element

c) value of p for nodes residing on Γ_1

d) value of γ, q for each segments with nodes on Γ_2

2. Elemental interpolation



If linear triangular elements are used,

$$\phi^e(x,y) = a^e + b^e x + c^e y$$

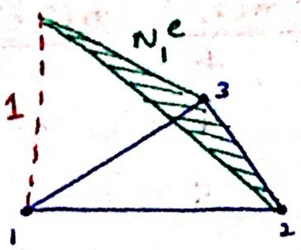
For each node in triangular element

$$\phi_i^e = a^e + b^e x_i^e + c^e y_i^e \quad ; \quad i = 1, 2, 3$$

$$\Rightarrow \phi^e(x,y) = \sum_{j=1}^{3} N_j^e(x,y)\,\phi_j^e$$

where $N_j^e(x,y) = \dfrac{1}{2\Delta^e}\left(a_j^e + b_j^e x + c_j^e y\right)$

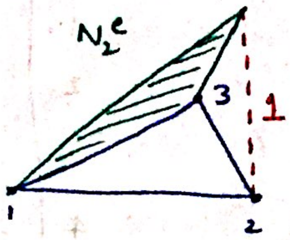
and $\Delta^e \equiv$ area of e^{th} element $= \dfrac{1}{2}\begin{vmatrix} 1 & x_1^e & y_1^e \\ 1 & x_2^e & y_2^e \\ 1 & x_3^e & y_3^e \end{vmatrix}$



$a_1^e = x_2^e y_3^e - y_2^e x_3^e$

$b_1^e = y_2^e - y_3^e$

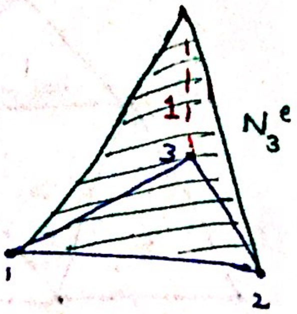
$c_1^e = x_3^e - x_2^e$



$a_2^e = x_3^e y_1^e - y_3^e x_1^e$

$b_2^e = y_3^e - y_1^e$

$c_2^e = x_2^e - x_1^e$



$a_3^e = x_1^e y_2^e - y_1^e x_2^e$

$b_3^e = y_1^e - y_2^e$

$c_3^e = x_2^e - x_1^e$

3. Formulation via

A: Ritz Method

consider homogenous Neumann b.c. $\gamma = q = 0$

STEP I
$$F(\phi) = \sum_{e=1}^{M} F^e(\phi^e)$$

where $M \equiv$ total no. of elements

$$F^e(\phi^e) = \frac{1}{2} \iint_{\Omega^e} \left[\alpha_x \left(\frac{\partial \phi^e}{\partial x} \right)^2 + \alpha_y \left(\frac{\partial \phi^e}{\partial y} \right)^2 + \beta (\phi^e)^2 \right] d\Omega$$
$$- \iint_{\Omega^e} f \phi^e \, d\Omega$$

where $\Omega^e \equiv$ domain of e^{th} element.

Now, $$\frac{\partial F^e}{\partial \phi_i^e} = \sum_{j=1}^{3} \phi_j^e \iint_{\Omega^e} \left(\alpha_x \frac{\partial N_i^e}{\partial x} \frac{\partial N_j^e}{\partial x} + \alpha_y \frac{\partial N_i^e}{\partial y} \frac{\partial N_j^e}{\partial y} + \beta N_i^e N_j^e \right) d\Omega$$
$$- \iint_{\Omega^e} f N_i^e \, d\Omega \, . \qquad i = 1,2,3$$

$$\Rightarrow \left\{ \frac{\partial F^e}{\partial \phi^e} \right\} = [K^e]\{\phi^e\} - \{b^e\}$$

where $$\left\{ \frac{\partial F^e}{\partial \phi^e} \right\} = \left[\frac{\partial F^e}{\partial \phi_1^e} \quad \frac{\partial F^e}{\partial \phi_2^e} \quad \frac{\partial F^e}{\partial \phi_3^e} \right]^T$$
$$\{\phi^e\} = [\phi_1^e \quad \phi_2^e \quad \phi_3^e]^T$$

and
$$K_{ij}^e = \iint_{\Omega^e} \left(\alpha_x \frac{\partial N_i^e}{\partial x} \frac{\partial N_j^e}{\partial x} + \alpha_y \frac{\partial N_i^e}{\partial y} \frac{\partial N_j^e}{\partial y} + \beta N_i^e N_j^e \right) dx\,dy$$

symmetric matrix ↑

$i, j = 1,2,3$

$$b_i^e = \iint_{\Omega^e} f N_i^e \, dx\, dy \quad ; \quad i = 1,2,3$$

Assuming $\alpha_x, \alpha_y, \beta, f$ are constants within each element and equal to $\alpha_x^e, \alpha_y^e, \beta^e, f^e$

$$\Rightarrow K_{ij}^e = \frac{1}{4\Delta^e}\left(\alpha_x^e b_i^e b_j^e + \alpha_y^e c_i^e c_j^e\right) + \frac{\Delta^e}{12}\beta^e (1+\delta_{ij})$$

$$b_i^e = \frac{\Delta^e}{3} f^e$$

If $\alpha_x, \alpha_y, \beta, f$ are not constant within each element, we can use above result with $\alpha_x^e, \alpha_y^e, \beta^e, f \equiv$ average values within the element.

NOTE: To evaluate integrals for K_{ij}^e, b_i^e we use formula:

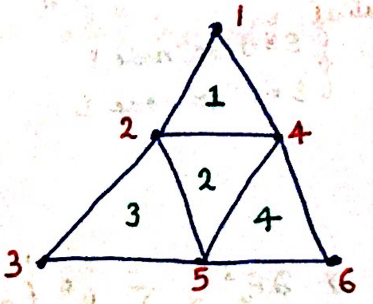
$$\iint_{\Omega^e} (N_1^e)^l (N_2^e)^m (N_3^e)^n \, dx\, dy = \frac{l!\, m!\, n!}{(l+m+n+2)!} 2\Delta^e$$

STEP 2

$$\left\{\frac{\partial F}{\partial \phi}\right\} = \sum_{e=1}^{M} \left\{\frac{\partial F^e}{\partial \phi^e}\right\} = \sum_{e=1}^{M} \left([\bar{K}^e]\{\bar{\phi}^e\} - \{\bar{b}^e\}\right) = \{0\}$$

NOTE: Overbar denotes augmented/expanded vector/matrix.

$$\Rightarrow [K]\{\phi\} = \{b\}$$



example:

$$[K] = \sum_{e=1}^{M} [\bar{K}^e]$$

Now, $$[\bar{K}^{(1)}] = \begin{bmatrix} K_{33}^{(1)} & K_{31}^{(1)} & 0 & K_{32}^{(1)} & 0 & 0 \\ K_{13}^{(1)} & K_{11}^{(1)} & 0 & K_{12}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ K_{23}^{(1)} & K_{21}^{(1)} & 0 & K_{22}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

e	n(1,e)	n(2,e)	n(3,e)
1	2	4	1
2	5	4	2
3	3	5	2
4	5	6	4

General rule: $K_{ij}^e \longrightarrow K_{n(i,e),n(j,e)}$

Similarly we can find out other $[\bar{K}^{(e)}]$ and add them all to get $[K]$

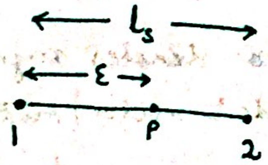
$$[K] = \left[\begin{array}{c|c|c|c|c|c}
K_{33}^{(1)} & K_{31}^{(1)} & 0 & K_{32}^{(1)} & 0 & 0 \\
K_{13}^{(1)} & K_{11}^{(1)}+K_{33}^{(2)}+K_{33}^{(3)} & K_{31}^{(3)} & K_{12}^{(1)}+K_{32}^{(2)} & K_{31}^{(2)}+K_{32}^{(3)} & 0 \\
0 & K_{13}^{(3)} & K_{11}^{(3)} & 0 & K_{12}^{(3)} & 0 \\
K_{23}^{(1)} & K_{21}^{(1)}+K_{23}^{(2)} & 0 & K_{22}^{(1)}+K_{22}^{(2)}+K_{33}^{(4)} & K_{21}^{(2)}+K_{31}^{(4)} & K_{32}^{(4)} \\
0 & K_{13}^{(2)}+K_{23}^{(3)} & K_{21}^{(3)} & K_{12}^{(2)}+K_{13}^{(4)} & K_{11}^{(2)}+K_{22}^{(3)}+K_{11}^{(4)} & K_{12}^{(4)} \\
0 & 0 & 0 & K_{23}^{(4)} & K_{21}^{(4)} & K_{22}^{(4)}
\end{array}\right]$$

Similarly we can assemble $\{b\}$ by adding $b_i^e \longrightarrow b_{n(i,e)}$

$$\{b\} = \left\{\begin{array}{c}
b_3^{(1)} \\
b_1^{(1)}+b_3^{(2)}+b_3^{(3)} \\
b_1^{(3)} \\
b_2^{(1)}+b_2^{(2)}+b_3^{(4)} \\
b_1^{(2)}+b_2^{(3)}+b_1^{(4)} \\
b_2^{(4)}
\end{array}\right\}$$

NOTE: Suppose we consider general b.c on Γ_2

extra term $\longrightarrow$ $F_b(\phi) = \int_{\Gamma_2} \left(\frac{\gamma}{2}\phi^2 - q\phi\right) d\Gamma$



$$= \sum_{s=1}^{M_s} F_b^s(\phi^s)$$

where $M_s \equiv$ element sides or segments

$\uparrow$ integral over segment s.

normalised func. $\begin{cases} \varepsilon = 0 \text{ at node 1} \\ \varepsilon = 1 \text{ at node 2} \end{cases}$

where $\phi^s \simeq \sum_{j=1}^{2} N_j^s \phi_j^s$

with $N_1^s = 1 - \varepsilon$, $N_2^s = \varepsilon$ (??)

$$\Rightarrow \frac{\partial F_b^s}{\partial \phi_i^s} = \sum_{j=1}^{2} \phi_j^s \int_0^1 \gamma N_i^s N_j^s L^s \, d\varepsilon - \int_0^1 q N_i^s L^s \, d\varepsilon$$

$$\Rightarrow \left\{\frac{\partial F_b^s}{\partial \phi^s}\right\} = [K^s] \{\phi^s\} - \{b^s\}$$

with $K_{ij}^s = \int_0^1 \gamma N_i^s N_j^s L^s \, d\varepsilon$; $i, j = 1, 2$

$b_i^s = \int_0^1 q N_i^s L^s \, d\varepsilon$; $i = 1, 2$

If $\gamma, q \equiv$ constants within each segment

$$K_{ij}^{s} = \gamma^{s} \frac{L^{s}}{6} (1+\delta_{ij})$$

$$b_i^{s} = q^{s} \frac{L^{s}}{2}$$

Now, to include F_b into the system:

$$\left\{\frac{\partial F}{\partial \phi}\right\} = \sum_{e=1}^{M} \left\{\overline{\frac{\partial F^e}{\partial \phi^e}}\right\} + \sum_{s=1}^{M_s} \left\{\overline{\frac{\partial F_b^s}{\partial \phi^s}}\right\}$$

$$= \sum_{e=1}^{M} \left([\bar{K}^e]\{\bar{\phi}^e\} - \{\bar{b}^e\}\right) + \sum_{s=1}^{M_s} \left([\bar{K}^s]\{\bar{\phi}^s\} - \{\bar{b}^s\}\right) = \{0\}$$

$\uparrow$

For augmentation, we need arrays that relates segments and global numbers of associated nodes.

Denoted by $ns(i,s)$ where $i = 1,2$
$s = 1,2,\ldots, M_s$

$\Rightarrow$ $[K^s]$ can be included in $[K]$ by adding each $K_{ij}^{s} \longrightarrow K_{ns(i,s)\, ns(j,s)}$

and,

$\{b^s\}$ can be included in $\{b\}$ by adding each $b_i^s \longrightarrow b_{ns(i,s)}$

$\boxed{\text{STEP-3}}$ Assume nodes 3, 5, 6 are on Γ_1

$\Rightarrow \phi_3 = p_3, \ \phi_5 = p_5, \ \phi_6 = p_6$

To impose this condn:

ϕ_3: $K_{33} = 1$; $K_{3i} = 0$ for $i = 1,2,4,5,6.$; $b_3 = p_3$

Similar to that in 1-D, to restore symmetry

$b_i \leftarrow b_i - K_{i3} p_3$, $K_{i3} = 0$ for $i = 2,4,5,1,6$

We can do the same for ϕ_5, ϕ_6

$$\Rightarrow [K] = \begin{bmatrix} K_{11} & K_{12} & 0 & K_{14} & 0 & 0 \\ K_{21} & K_{22} & 0 & K_{24} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ K_{41} & K_{42} & 0 & K_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and,

$$\{b\} = \begin{Bmatrix} b_1 - K_{13} p_3 - K_{15} p_5 - K_{16} p_6 \\ b_2 - K_{23} p_3 - K_{25} p_5 - K_{26} p_6 \\ p_3 \\ b_4 - K_{43} p_3 - K_{35} p_5 - K_{48} p_6 \\ p_5 \\ p_6 \end{Bmatrix}$$

$\Rightarrow$ Resultant system of equations:

$$\begin{bmatrix} K_{11} & K_{12} & K_{14} \\ K_{21} & K_{22} & K_{24} \\ K_{41} & K_{42} & K_{44} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} b_1 - K_{13} p_3 - K_{15} p_5 - K_{16} p_6 \\ \vdots \\ \end{Bmatrix}$$

Alternate method

Instead of putting $\phi_3 = p_3$ we choose large number (say 10^{70})

to set: $K_{33} = 10^{70}$

$b_3 = p_3 \times 10^{70}$

making eq^n associated with $\phi_3 \equiv \phi_3 = p_3$, provided no other matrix elements and unknowns are of similar order to 10^{70}.

$$\Rightarrow \begin{bmatrix} K_{11} & K_{12} & \cdots & & & K_{16} \\ K_{21} & \cdots & & & & K_{26} \\ K_{31} & K_{32} & 10^{70} & K_{34} & K_{35} & K_{36} \\ K_{41} & \cdots & & & & K_{46} \\ K_{51} & \cdots & & K_{54} & 10^{70} & K_{56} \\ K_{61} & \cdots & & K_{64} & K_{65} & 10^{70} \end{bmatrix} \begin{Bmatrix} \phi \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ p_3 \times 10^{70} \\ b_4 \\ p_5 \times 10^{70} \\ p_6 \times 10^{70} \end{Bmatrix}$$

For general problem: having N_1 nodes on Γ_1

$K_{nd(i),\, nd(i)} = 10^{70}$, $b_{nd(i)} = p(i) \times 10^{70}$; $i = 1, 2, 3, \ldots, N_1$

$\uparrow$

This method does not permit reduction / elimination of eq^{n}s.

$\boxed{\text{B: Galerkin's method}}$

$\boxed{\text{STEP-1}}$

$$r = -\frac{\partial}{\partial x}\left(\alpha_x \frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\alpha_y \frac{\partial \phi}{\partial y}\right) + \beta\phi - f$$

Weighted residual for element e :

$$R_i^e = \iint_{\Omega^e} N_i^e \, r \, dx dy \quad ; \; i = 1, 2, 3$$

On simplification :

$$R_i^e = \iint_{\Omega^e} \Big(\underbrace{\alpha_x \frac{\partial N_i^e}{\partial x}\frac{\partial \phi}{\partial x} + \alpha_y \frac{\partial N_i^e}{\partial y}\frac{\partial \phi}{\partial y} + \beta N_i^e \phi}_{} - \underbrace{N_i^e f}_{} \Big) dx\, dy$$

$$- \oint_{\Gamma^e} \underbrace{N_i^e \, \bar{D} \cdot \hat{n}^e}_{} \, d\Gamma$$

$$\bar{D} = \alpha_x \frac{\partial \phi}{\partial x} \hat{x} + \alpha_y \frac{\partial \phi}{\partial y} \hat{y}$$

$\hat{n}^e \equiv$ outward unit vector normal to Γ^e

$$= \sum_{j=1}^{3} \left(\alpha_x \frac{\partial N_i^e}{\partial x}\frac{\partial N_j^e}{\partial x} + \alpha_y \frac{\partial N_i^e}{\partial y}\frac{\partial N_j^e}{\partial y} + \beta N_i^e N_j^e \right) \Phi_j \, dx\, dy$$

$$- \iint_{\Omega^e} N_i^e f \, dx dy \; - \oint_{\Gamma^e} N_i^e \, \bar{D} . \hat{n}^e \, d\Gamma$$

$$\Rightarrow \{R^e\} = [K^e]\{\phi^e\} - \{b^e\} - \{g^e\}$$

where $\{g^e\} := g_i^e = \int_{\Gamma^e} N_i^e \, \bar{D} . \hat{n}^e \, d\Gamma$

$\boxed{\text{STEP-2}}$

For all elements :

$$\{R\} = \sum_{e=1}^{M} \{\bar{R}^e\} = \sum_{e=1}^{M} \left([\bar{K}^e]\{\bar{\Phi}^e\} - \{\bar{b}^e\} - \{\bar{g}^e\} \right)$$

$$\Rightarrow [K]\{\phi\} = \{b\} + \underset{\uparrow}{\{g\}}$$

should be absorbed in [K] and {b} as Galerkin's and Ritz method should yield same result.

i.e.

$$\{g\} = \begin{Bmatrix} g_3^{(1)} \\ g_1^{(1)} + g_3^{(2)} + g_3^{(3)} \\ g_1^{(3)} \\ g_2^{(1)} + g_2^{(2)} + g_3^{(4)} \\ g_1^{(2)} + g_2^{(3)} + g_1^{(4)} \\ g_2^{(4)} \end{Bmatrix}$$

NOTE: N_i^e vanishes at element side opposite to node i

As $N_1^{(1)} = N_3^{(2)}$, $\hat{n}^{(1)} = -\hat{n}^{(2)}$ on element sides connecting 2↔4

$$g_2 = \left[\int_{\text{node}1}^{\text{node }2} + \cancel{\int_2^4} \right] N_1^{(1)} \bar{D} . \hat{n}^{(1)} \, d\Gamma + \left[\cancel{\int_2^5} + \cancel{\int_4^2} \right] N_3^{(2)} \bar{D} . \hat{n}^{(2)} \, d\Gamma$$

$$+ \left[\int_2^3 + \cancel{\int_5^2} \right] N_3^{(3)} \bar{D} . \hat{n}^{(3)} \, d\Gamma$$

$$= \int_1^2 N_1^{(1)} \bar{D} . \hat{n}^{(1)} \, d\Gamma + \int_2^3 N_3^{(3)} \bar{D} . \hat{n}^{(3)} \, d\Gamma$$

Similarly

$$g_4 = \int_4^1 N_2^{(1)} \bar{D} . \hat{n} \, d\Gamma + \int_6^4 N_3^{(4)} \bar{D} . \hat{n} \, d\Gamma$$

$$g_5 = \int_3^5 N_2^{(3)} \bar{D} . \hat{n} \, d\Gamma + \int_5^6 N_1^{(4)} \bar{D} . \hat{n} \, d\Gamma$$

OBSERVE: Inertial element sides do not contribute to $\{g\}$; only those residing on boundary Γ contribute.

i.e.

$g_i = 0 \quad \forall \; i \in$ node resides inside Γ

$\neq 0 \quad \forall \; i \in$ node resides on boundary Γ

$$\Rightarrow g_i = \underset{\substack{\uparrow \\ \text{segment to} \\ \text{right of node } i}}{\int_{\Gamma^s}} N_j^e D . \hat{n} \, d\Gamma + \underset{\substack{\uparrow \\ \text{segment to} \\ \text{left of node } i}}{\int_{\Gamma^{s+1}}} N_j^e \bar{D} . \hat{n} \, d\Gamma$$

NOTE: $i = n(j,e)$

STEP-3

If node resides on Γ_1 ← Derichlet condn exists.

$\Rightarrow$ g_i term needs to be discarded.

If node resides on Γ_2

(i) <u>homogenous Neumann b.c</u>

$$\Rightarrow g_i = 0 \quad \because \bar{D}.\hat{n} = 0$$

(ii) <u>b.c. of third kind</u>

$$g_i = \int_{\Gamma^s} N_j^e (q-\gamma\phi^s)\, d\Gamma + \int_{\Gamma^{s+1}} N_j^e (q-\gamma\phi)\, d\Gamma$$

Since N_j^s varies linearly from 1 at node i → 0 at neighbouring nodes

$$\Rightarrow g_i = \int_0^1 \varepsilon (q-\gamma\phi^s) L^s\, d\varepsilon + \int_0^1 (1-\varepsilon)(q-\gamma\phi^{s+1}) L^{s+1}\, d\varepsilon$$

<u>NOTE</u>: Using expn. $\phi^s = \sum_{j=1}^{2} N_j^s \phi_j^s$ where $N_1^s = 1-\varepsilon$

$N_2^s = \varepsilon$

and $\varepsilon \equiv$ normalised distance measured from node 1 to node 2 on segment

$L^s \equiv$ distance / length of segment

$$\Rightarrow g_i = \int_0^1 N_2^s \left(q - \gamma \sum_{j=1}^{2} N_j^s \phi_j^s\right) L^s\, d\varepsilon + \int_0^1 N_1^{s+1}\left(q - \gamma \sum_{j=1}^{2} N_j^{s+1} \phi_j^{s+1}\right) L^{s+1}\, d\varepsilon$$

$$\Rightarrow g_i = \underset{\substack{\uparrow \\ \text{add to } b_i}}{b_2^s} - \underset{\substack{\uparrow \\ \text{add to } K_{i,ns(j,s)}}}{\sum_{j=1}^{2} K_{2j}^s \phi_j^s} + \underset{\substack{\uparrow \\ \text{add to } b_i}}{b_1^{s+1}} - \underset{\substack{\uparrow \\ \text{add to } K_{i,ns(j,s+1)}}}{\sum_{j=1}^{2} K_{1j}^{s+1} \phi_j^{s+1}}$$

4. Solution of System of equations

<u>Direct methods</u>

↳ Gaussian elimination

↓

yields exact solⁿ

<u>Indirect methods</u>

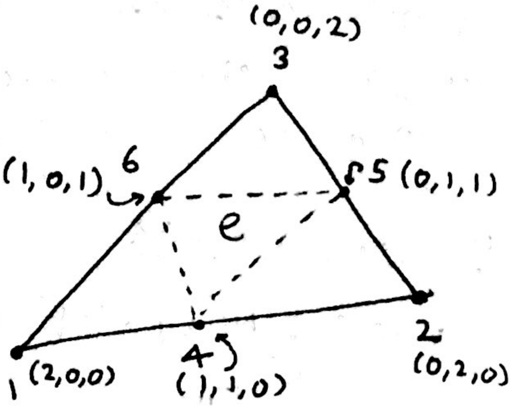
↳ iterative : hit and trial method, minimize residual error

↓

yields approx. solⁿ with prescribed accuracy

* Higher order elements

I. Quadratic Triangular Elements



$$\phi^e(x,y) = a^e + b^e x + c^e y + d^e x^2 + e^e xy + f^e y^2$$

$$= \sum_{j=1}^{6} N_j^e \, \phi_j^e$$

where $N_j^e(x,y) = (2L_j^e - 1)L_j^e$; $j = 1,2,3$

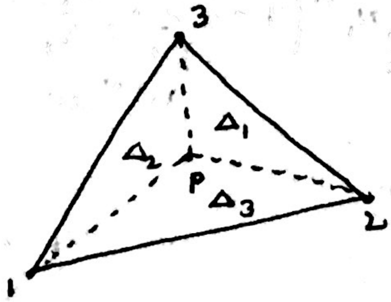
$N_4^e = 4L_1^e L_2^e$, $N_5^e = 4L_2^e L_3^e$, $N_6^e = 4L_3^e L_1^e$

with $L_j^e(x,y) = \frac{1}{2\Delta^e}(a_j^e + b_j^e x + c_j^e y)$; $j = 1,2,3$

OBSERVE: $N_i^e = \delta_{ij}$

Construction of interpolation functions

i.e. Lagrangian's interpolation polynomials



Consider point P in triangular element.

$$\Delta_1 = \frac{1}{2}\begin{vmatrix} 1 & x & y \\ 1 & x_2^e & y_2^e \\ 1 & x_3^e & y_3^e \end{vmatrix}$$

$$= \frac{1}{2}\left[(x_2^e y_3^e - x_3^e y_2^e) + x(y_2^e - y_3^e) + y(x_3^e - x_2^e)\right]$$

$$= \frac{1}{2}(a_1^e + b_1^e x + c_1^e y)$$

$$\Rightarrow L_i^e = \frac{\Delta_i}{\Delta^e} = \frac{1}{2\Delta^e}(a_i^e + b_i^e x + c_i^e y)$$

i.e. $$\begin{Bmatrix} L_1^e \\ L_2^e \\ L_3^e \end{Bmatrix} = \frac{1}{2\Delta^e}\begin{bmatrix} a_1^e & b_1^e & c_1^e \\ a_2^e & b_2^e & c_2^e \\ a_3^e & b_3^e & c_3^e \end{bmatrix}\begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$; $i = 1,2,3$

NOTE: Choice of pt P completely defines L_i^e, so one can conversely use them to define a pt in Δ^e.

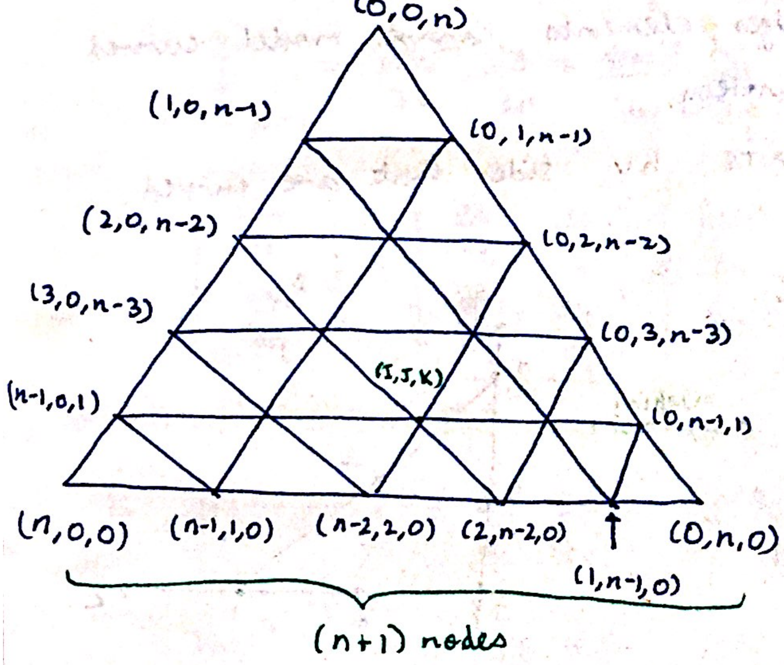
ie.
$$\begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1^e & x_2^e & x_3^e \\ y_1^e & y_2^e & y_3^e \end{bmatrix} \begin{Bmatrix} L_1^e \\ L_2^e \\ L_3^e \end{Bmatrix}$$

(L_1^e, L_2^e, L_3^e) ≡ area coordinates
natural "
simplex "

II. General triangular element

To guarantee sol^n isotropy, polynomial expression ≡ order n.
ie. contain all possible terms $x^p y^q$, $0 \le p+q \le n$
ie. no. of terms, $m = \frac{1}{2}(n+1)(n+2)$

⇒ To express m coeff of polynomial ⇒ m nodes in element



Interpolation function

$$N_i^e = P_I^n(L_1^e)\, P_J^n(L_2)\, P_K^n(L_3)$$
; $I+J+K = n$

where:
$$P_I^n(L_1^e) = \prod_{p=0}^{I-1} \frac{nL_1^e - p}{I-p}$$
$$= \frac{1}{I!} \prod_{p=0}^{I-1} (nL_1^e - p)$$

with
$$P_0^n = 1$$

OBSERVE: At node (I, J, K)
$$L_1^e = \frac{I}{n}, \quad L_2^e = \frac{J}{n}, \quad L_3^e = \frac{K}{n}$$
$\Rightarrow P_I^n = P_J^n = P_K^n = 1$
$\Rightarrow N_i^e = 1$

At any other node :
$$\left.\begin{array}{l} P_{I'}^n(I/n) = 0, \ I' > I \\ \text{or } P_{J'}^n(J/n) = 0, \ J' > J \\ \text{or } P_{K'}^n(K/n) = 0, \ K' > K \end{array}\right\} N_i^e = 0$$

Numerical integration

If $\alpha_x, \alpha_y, \beta, f$ associated with each element are not consta we employ interpolation func^n to expand these parameters in terms of their nodal values and then evaluate analytica

$$\boxed{\iint_{\Omega^e} F(L_1^e, L_2^e, L_3^e)\, dx dy = \sum_{i=1}^{m} W_i\, F(L_{1i}^e, L_{2i}^e, L_{3i}^e)\, \Delta^e}$$

GAUSSIAN QUADRATURE

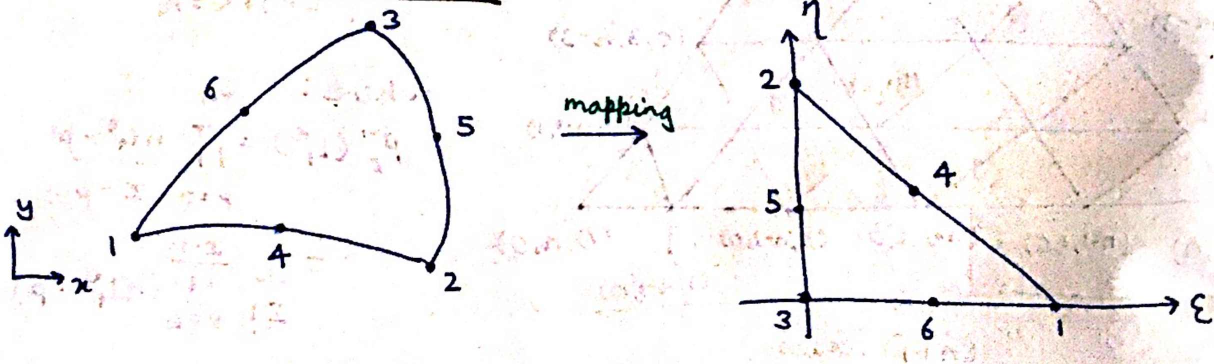
★ Isoparametric elements

→ major advantage of higher-order elements ≡ for specified accuracy, they can use larger elements & hence, smaller no. of unknowns.

→ However, straight-edges elements cannot model curved boundaries accurately.

ISOPARAMETRIC ELEMENTS hv sides that are curved.

I. Triangular Elements



$$x = a + b\varepsilon + c\eta + d\varepsilon^2 + e\,\varepsilon\eta + f\eta^2$$
$$y = a' + b'\varepsilon + c'\eta + d'\varepsilon^2 + e'\,\varepsilon\eta + f'\eta^2$$

or

$$x = \sum_{j=1}^{6} N_j^e(\varepsilon, \eta)\, x_j$$

$$y = \sum_{j=1}^{6} N_j^e(\varepsilon, \eta)\, y_j$$

, where :

$$N_i^e = P_I^n(1-\varepsilon-\eta)\, P_J^n(\varepsilon)\, P_K^n(\eta),$$
$$I+J+K = n$$

After triangular element is mapped into ε-η plane:

$$\phi^e(\varepsilon,\eta) = \sum_{j=1}^{6} N_j^e(\varepsilon,\eta)\,\phi_j^e$$

<u>NOTE</u>: Interpolation functions do not have to be of same order as geometrical transformation.

If they are, elements are called <u>ISOPARAMETRIC</u>

else they are called subparametric (or superparametric)

$$K_{ij}^e = \iint_{\Omega^e} \left(\alpha_x \frac{\partial N_i^e}{\partial x}\frac{\partial N_j^e}{\partial x} + \alpha_y \frac{\partial N_i^e}{\partial y}\frac{\partial N_j^e}{\partial y} + \beta N_i^e N_j^e \right) dx\,dy$$

$$b_i^e = \iint_{\Omega^e} f N_i^e \,dx\,dy$$

Now,
$$\frac{\partial N_i^e}{\partial \varepsilon} = \frac{\partial N_i^e}{\partial x}\frac{\partial x}{\partial \varepsilon} + \frac{\partial N_i^e}{\partial y}\frac{\partial y}{\partial \varepsilon}$$

$$\frac{\partial N_i^e}{\partial \eta} = \frac{\partial N_i^e}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial N_i^e}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\Rightarrow \begin{Bmatrix} \frac{\partial N_i^e}{\partial \varepsilon} \\ \frac{\partial N_i^e}{\partial \eta} \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \varepsilon} & \frac{\partial y}{\partial \varepsilon} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}}_{[J]:\ \text{jacobian matrix}} \begin{Bmatrix} \frac{\partial N_i^e}{\partial x} \\ \frac{\partial N_i^e}{\partial y} \end{Bmatrix}$$

$$\Rightarrow \frac{\partial N_i^e}{\partial x} = \frac{1}{|J|}\left(\frac{\partial y}{\partial \eta}\frac{\partial N_i^e}{\partial \varepsilon} - \frac{\partial y}{\partial \varepsilon}\frac{\partial N_i^e}{\partial \eta} \right)$$

$$\frac{\partial N_i^e}{\partial y} = \frac{1}{|J|}\left(-\frac{\partial x}{\partial \eta}\frac{\partial N_i^e}{\partial \varepsilon} + \frac{\partial x}{\partial \varepsilon}\frac{\partial N_i^e}{\partial \eta} \right)$$

and

$$dx\,dy = |J|\,d\varepsilon\,d\eta$$

Substitute above in elemental equations.

They can be evaluated using Gaussian quadrature.

$$\begin{cases} K_{ij}^e = \int_0^1\int_0^{1-\eta} \left(\qquad \right) |J|\,d\varepsilon\,d\eta \\ b_i^e = \int_0^1\int_0^{1-\eta} f N_i^e\,|J|\,d\varepsilon\,d\eta \end{cases}$$

ALTERNATE METHOD: Gauss - Legendre quadrature formula:

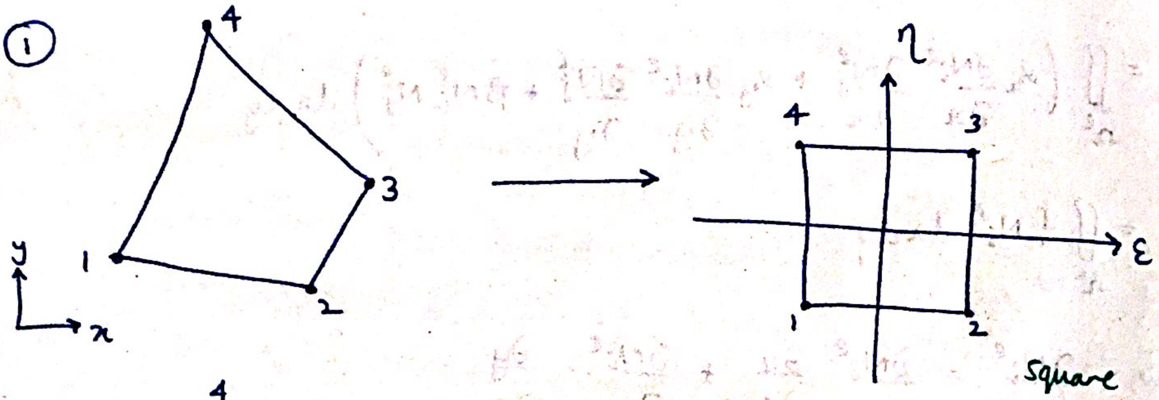
$$\varepsilon' = \frac{2\varepsilon}{1-\eta} - 1 \quad , \eta' = 2\eta - 1$$

$$\int_0^1 \int_0^{1-\eta} p(\varepsilon,\eta)\, d\varepsilon\, d\eta = \int_{-1}^{1}\int_{-1}^{1} \underset{\uparrow}{\tilde{p}}(\varepsilon',\eta')\, d\varepsilon'\, d\eta'$$

$$= \frac{1}{4}(1-\eta)\,p(\varepsilon,\eta)$$

II. Quadrilateral elements

①



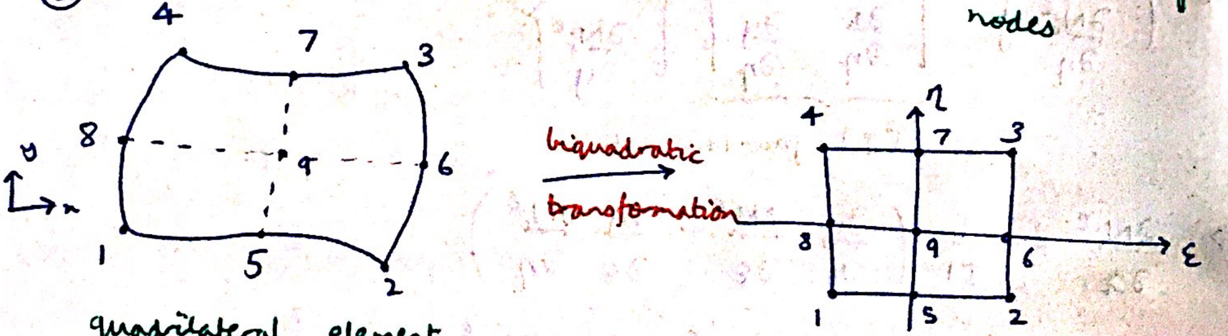
$$x = \sum_{j=1}^{4} N_j^e(\varepsilon,\eta)\, x_j$$

$$y = \sum_{j=1}^{4} N_j^e(\varepsilon,\eta)\, y_j$$

, where $\underset{\uparrow}{N_j^e}(\varepsilon,\eta) = \frac{1}{4}(1+\varepsilon_j\varepsilon)(1+\eta_j$

$(\varepsilon_j, \eta_j) \equiv$ coordinates of the nodes

②



quadrilateral element with sides defined by second-order elements

$$N_j^e(\varepsilon,\eta) = \underset{\uparrow}{N_j^e(\varepsilon)}\, N_j^e(\eta)$$

one-D Langragian interpolation polynomial

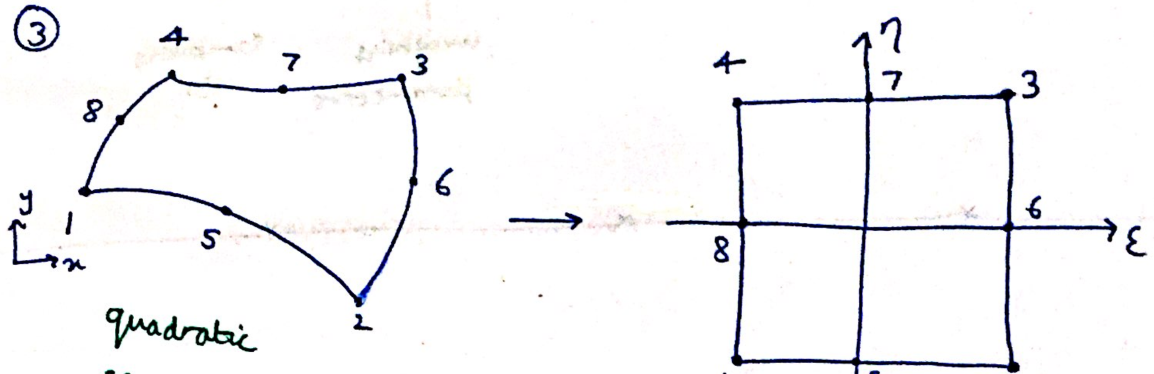
However, this type of element is not very useful as there are incomplete interpolation functions.

REASONS :

i) A fourth order polynomial ≡ 15 interpolation functions but we hv only 9.

ii) Interpolation functions are restrictive i.e. vanish along all four lines $\varepsilon, \eta = \{0, 1\}$.

③



quadratic serendipity

↑

most useful isoparametric quadrilateral element.

In this case, interpolation functions are constructed s.t. they vanish along two lines only.

Example : $N_1^e(\varepsilon,\eta) \equiv$ vanishes along $\varepsilon = 1, \eta = 1$

$$\Rightarrow N_1^e(\varepsilon,\eta) = (\underset{\uparrow}{a} + \underset{\uparrow}{b}\varepsilon + \underset{\uparrow}{c}\eta)(1-\varepsilon)(1-\eta)$$

Now, at node 1 : $N_1^e = 1$
node 5, 8 : $N_1^e = 0$

$$\left.\begin{matrix} \\ \end{matrix}\right\} a = b = c = -\frac{1}{4}$$

Similarly other interpolation functions can be constructed.

$$N_j^e = \frac{1}{4}(1+\varepsilon_j\varepsilon)(1+\eta_j\eta)(\varepsilon_j\varepsilon + \eta_j\eta - 1) \; ; \; j = 1,2,3,4$$

$$N_j^e = \frac{1}{2}(1+\varepsilon)(1-\varepsilon)(1+\eta_j\eta) \; ; \; j = 5,7$$

$$N_j^e = \frac{1}{2}(1+\varepsilon_j\varepsilon)(1+\eta)(1-\eta) \; ; \; j = 6,8$$

OBSERVE: Highest order of function = 3

A complete third order polyn. → 10 interpolation functions

⇒ interpolation defined above is more complete than Lagrangian element.

Elemental matrix and vector can be evaluated using one-dimensional Gauss - Legendre quadrature: in each dimension:

$$\int_{-1}^{1}\int_{-1}^{1} P(\varepsilon,\eta)\, d\varepsilon\, d\eta = \sum_{i=1}^{n}\sum_{j=1}^{n} \underbrace{W_i W_j}_{\substack{\uparrow \\ \text{weighing} \\ \text{parameters}}} P(\underbrace{\varepsilon_i,\eta_j}_{\substack{\uparrow \\ \text{sampling} \\ \text{pts}}})$$